UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

Representation of Non-Commutative Topological Algebras

by

S. M. Hacking.

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I N D E X

INTRODUCTION

The well known Gelfand-Naimark theorem enables us to represent a complex commutative C*-algebra as a full algebra of complex valued functions defined on its set of primitive ideals which is called the structure space of the algebra. In this thesis we are concerned with the generalization of this type of representation theorem to non-commutative rings and algebras. In order to prove the Gelfand-Naimark theorem, we needed the Stone-Weierstrass theorem to enable us to show that a subalgebra is actually equal to a full algebra of functions. We shall see that in order to represent a non-commutative algebra as a set of functions taking values in a variable range, we shall need a suitable type of Stone-Weierstrass theorem. This thesis can therefore be considered as an illustration of the application of Stone-Weierstrass type arguments to the theory of C*-algebra representations.

Chapter I is introductory in nature and considers the representation of real and complex commutative algebras. Any results which are proved in text books will only be stated with no proof being given. The set of complex algebras can be considered as a subset of the set of real algebras and we shall see that there are two different approaches to the representation of a real algebra. Either we can try to extend the type of results that are known for complex algebras or embed the real algebra in its complexification. R.V. Kadison [16] approached this problem using the first method. Unfortunately his paper is too long to give the proof of Theorem 1.2.17 which shows us under which conditions a real Banach algebra can be represented as a full algebra of real valued functions.

The second approach was followed by R. Arens and I. Kaplansky [1] who by considering the complexification of a real B*-algebra represented it as an algebra of functions which may be either real or complex valued. See Theorem 1.2.21.

Finally Theorem 1.2.8 shows us what conditions are necessary for a real B*-algebra to be isometrically-*-isomorphic to a C*-algebra. In general a real B*-algebra is not isometrically-*-isomorphic to a C*-algebra as C*-algebras are necessarily symmetric and this is not the case with real B*-algebras. The proof of Theorem 1.2.8 does not use the complexification of the algebra whereas L. Ingelstam [14] proved a similar result using the complexification.

Chapter II is concerned with the representation of non-commutative topological algebras by functions with some sort of continuity property. The first advances were made by

I. Kaplansky [23] who considered functions or vector fields σ defined on a locally compact Hausdorff space T taking values $\sigma(t)$ in Banach algebras A_t. That is $\sigma \in \prod_{t \in t} A_t$. The continuity property of these functions is that the map $t \to \| \sigma(t) \|$ is continuous and σ will then be said to be norm-continuous. We cannot talk about continuity of the functions as there is no topology in the range.

Theorem 2.1.12 shows us that there is a relationship between the norm continuity of the functions and the Hausdorffness of the base space T, where T is the structure space of the algebra. Using the concept of a continuous field of Banach spaces a generalization of J. Glimm's [11] Stone-Weierstrass theorem can be proved, (Corollary 2.3.16) and hence the representation theorem, (Theorem 2.3.19) for G.C.R. and C.C.R. algebras, which have the property of having a composition series whose factors have a Hausdorff structure space.

In Chapter III we introduce a topology on the range space $B = \bigcup_{t \in t} A_t$ where the A_t are now C*-algebras. We shall be restricted to considering rings or algebras which have Hausdorff structure spaces and the concept of a vector field will be replaced by a section and we shall now be able to talk of continuity of the sections σ. Initially biregular rings are considered which have the very important property of having a locally compact, Hausdorff, totally disconnected structure space T . Using the theory of sheaves, Theorem 3.1.11 shows us that a biregular ring is isomorphic to a full set of sections with compact supports defined on T , the structure space.

We then consider homogeneous C*-algebras which have the property of having a Hausdorff structure space. We shall need the Stone-Weierstrass theorem of Chapter II in order to prove Theorem 3.2.4 which will enable us to prove the representation theorem, (Theorem 3.2.12) for homogeneous C*-algebras using the theory of fibre bundles.

Finally in Chapter IV the question of the representation of an arbitrary C*-algebra is considered. Unfortunately we shall not be able to give all the proofs of the theorems as the paper by J. Dauns and K.H. Hofmann [5] is rather long. The concept of a uniform field is introduced and a type of Stone-Weierstrass theorem, (Theorem 4.2.4) is stated which will then enable us to prove Theorem 4.3.6 the sought after generalization of the Gelfand-Naimark theorem for non-commutative C*-algebras. The base space will not be the structure space of the algebra but a completely regular space derived from the algebra as for a C*-algebra the structure space is not in general Hausdorff.

To keep the physical length of this survey within reasonable limits we had to omit the proofs of theorems which are readily available in the standard texts. Specific references are given in each case. For the same reason a few very lengthy proofs of results which have as yet only appeared in journal articles had to be left out aswell. This occured especially in Chapter IV which is essentially based on the extensive Memoir by J. Dauns and K. H. Hofmann [5]. In such instances an attempt was made however to outline the arguments leading to the main representation theorems.

Finally I should like to thank Dr. Kotzé, my supervisor, for the tremendous amount of encouragement and help he has given me, without which I feel this thesis would never have been written.

N O T A T I O N

The following symbols are used extensively in the text:

$\mathbb{N}$	will denote the set of natural numbers.
$\mathbb{R}$	will denote the set of real numbers.
$\mathbb{R}^+$	the set of positive real numbers.
$\mathbb{C}$	the set of complex numbers.
Φ	the empty set.
$C(X)$	the space of continuous complex valued functions defined on X.
$C^{\mathbb{R}}(X)$	the space of continuous real valued functions defined on X.
$C_o(X)$	the space of continuous complex valued functions vanishing at infinity defined on X.

The abbreviation "iff" is used for, "if and only if".

CHAPTER I

REPRESENTATION OF COMMUTATIVE ALGEBRAS

This is an introductory chapter about the representation of a commutative real or complex B*-algebra as an algebra of continuous real or complex valued functions defined on its structure space. Having shown that the algebra can be represented as a subalgebra of the full algebra of functions, an appropriate version of the Stone-Weierstrass theorem will be used to show that the subalgebra is in fact equal to the full algebra.

The first section contains definitions leading up to the concept of the structure space of an algebra. In the second section it will be shown that a real B*-algebra, whose spectrum satisfies a given property, is isometrically-*-isomorphic to a real C*-algebra (Theorem 1.2.8.) The question of the representation of a real Banach algebra is dealt with in this section and Theorem 1.2.17 shows under which conditions a real commutative Banach algebra can be represented as the full algebra of continuous real valued functions defined on a compact Hausdorff space, whereas Theorem 1.2.20 shows when a real commutative B*-algebra can be represented as an algebra of functions, which may be either real or complex defined on a different base space. Finally in the third section a representation theorem for a complex commutative B*-algebra will be stated (Theorem 1.3.6.)

Section 1. The Structure Space of an Algebra.

Let A be an algebra over a field F, χ a linear vector space over F and L (χ) the algebra of all the linear transformations of χ into itself.

1.1.1. Definition. A representation of A in L (χ) or on χ is any homomorphism of A into the algebra L (χ).

The representation is called faithful if the homomorphism is an isomorphism. If F is either the reals or complexes and χ a normed linear space, then a homomorphism of A into the algebra B (χ) of all bounded linear transformations of χ into itself is called a normed representation. In the case of a normed algebra, the term representation will mean normed representation. A representation of a normed algebra A on χ, that is a map $T : a \to T_a$ is said to be continuous or bounded if $\exists$ a constant β such that $\| T_a \| \leqslant \beta \| a \| \; \forall \, a \in A$.

1.1.2. Definition. A left regular representation of an algebra A is a homomorphism L from A into the algebra of linear transformations of the vector space A into itself defined by $L_a x = a x$, $\forall \, a, x \in A$.

If A is a normed algebra then
$\| L_a x \| \leqslant \| a \| \; \| x \| \Rightarrow \| L_a \| \leqslant \| a \|$. Hence the map $a \to L_a$ is a continuous normed representation of A.

1.1.3. Definition. A left ideal τ of an algebra A is said to be modular if $\exists\ y \in A$ such that $x - xy \in \tau$, $\forall\ x \in A$.

In an algebra A every modular ideal is contained in a maximal modular ideal and if A is a Banach algebra, then every maximal modular ideal is closed. Hence if A is a Banach algebra then the quotient A/M, where M is any maximal modular ideal is again a Banach algebra.

Let A be an algebra $\chi^{(1)}$, $\chi^{(2)}$ linear vector spaces and $T^{(1)}$, $T^{(2)}$ representations of A on $\chi^{(1)}$ and $\chi^{(2)}$ respectively.

1.1.4. Definition. $T^{(1)}$ and $T^{(2)}$ are algebraically equivalent, provided $\exists$ a one to one linear transformation U mapping $\chi^{(2)}$ onto $\chi^{(1)}$ such that $T_a^{(1)} U = U T_a^{(2)}$, $\forall\ a \in A$. If we are dealing with normed representations and U is a homeomorphism between the normed linear spaces $\chi^{(1)}$ and $\chi^{(2)}$ then the two representations are said to be topologically equivalent.

1.1.5. Definition. A linear subspace V of χ is said to be invariant with respect to B a subalgebra of $L(\chi)$ if $T(V) \subseteq V$, $\forall\ T \in B$.

Let $z \in \chi$ and $\chi_z = \{ T_z ; T \in B \}$, B a subalgebra of $L(\chi)$.

1.1.6. Definition. If $\exists\ z \in \chi$ such that $\chi_z = \chi$, then B is said to be <u>strictly cyclic</u> and if in the normed case $\bar{\chi}_z = \chi$, that is the closure of χ_z equals χ , then B is said to be <u>topologically cyclic</u>. In the first case z is called a strictly cyclic vector, and in the second a topologically cyclic vector.

1.1.7. <u>Definition</u>. B is said to be <u>strictly irreducible</u>, provided (0) and χ are the only invariant subspaces and in the normed case B is said to be <u>topologically irreducible</u>, provided (0) and χ are the only closed invariant subspaces.

Hence B is strictly (topologically) irreducible iff every non-zero vector in χ is strictly (topologically) cyclic.

Let τ be a left ideal in the algebra A and consider the representation $a \to L_a^\tau$ of A obtained by restricting to the ideal τ the left regular representation $a \to L_a$ of A . This representation is called the left regular representation on τ .

Now consider the representation $a \to L_a^{A/\tau}$ which is induced on the space A/τ by the left regular representation of A . That is $L_a^{A/\tau} x' = (L_a x)' = (a\ x)'$, $\forall\ x' \in A/\tau$. This representation is called the left regular representation on A/τ and the kernel of this representation is the set $\{k \in A\ ;\ k\ A \subseteq \tau\}$.

1.1.8. <u>Definition</u>. The <u>quotient</u> of the ideal τ will be denoted by $\tau : A = \{k \in A\ ;\ k\ A \subseteq \tau\}$. This is equal to the kernel of the left regular representation on A/τ .

We now come to the definition of the primitive ideal of an algebra.

1.1.9. Definition. An ideal P is said to be primitive if it is the quotient of a maximal modular left ideal.

1.1.10. Definition. An algebra A is said to be primitive if the zero ideal is primitive.

N. Jacobson [15] proves that an ideal in an algebra A is primitive whether A is considered to be an algebra or a ring, hence we have the following characterisation of a primitive ideal.

1.1.11. Proposition. In an algebra A the following statements are equivalent.

(i) P is a primitive ideal.

(ii) A/P is a primitive algebra.

(iii) P is the kernel of a strictly irreducible representation.

See N. Jacobson [15].

We have the following characterisation for a primitive algebra.

1.1.12. Proposition. An algebra A is primitive iff A has a faithful irreducible representation.

See C.E. Rickart [29].

We shall now state some very useful properties of primitive ideals.

1.1.13. Theorem. A an algebra then:

(i) Every maximal modular ideal is primitive.

(ii) P a primitive ideal and τ_1, τ_2 left ideals such that $\tau_1 \tau_2 \subseteq P$ then either $\tau_1 \subseteq P$ or $\tau_2 \subseteq P$.

(iii) If A is commutative then an ideal is primitive iff it is maximal modular.

(iv) When A is a Banach algebra, every primitive ideal is closed.

See C.E. Rickart [29].

The following is a characterisation of the radical which will be used in later chapters.

1.1.14. Definition. The radical R of an algebra A is equal to the intersection of the kernels of all strictly irreducible representations of A. If $R = (0)$ then A is said to be semi-simple and if $R = A$ then A is said to be a radical algebra.

1.1.15. Proposition.

(i) If A is not a radical algebra then R equals the intersection of all the primitive ideals of A which equals the intersection of all the maximal modular ideals of A.

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1.1.15. Proposition.

(i) If A is not a radical algebra then R equals the intersection of all the primitive ideals of A which equals the intersection of all the maximal modular ideals of A.

(ii) R is the minimal ideal containing all the left (right) ideals all of whose elements are left (right) quasi-regular. Hence there can exist no non-zero idempotent in the radical R of A.

See C.E. Rickart [29].

In the following Prim(A) will denote the set of primitive ideals of an algebra A and $M(A)$ the set of maximal modular ideals of A.

We shall now introduce a topology on Prim(A) called the hull-kernel topology.

1.1.16. Proposition. For T a subset of Prim(A) let $k(T)$ the kernel of T denote the intersection of the elements of T and for F a subset of A let $h(F)$ the hull of F denote the set of primitive ideals of A containing F. In terms of hulls and kernels a closure operation can be introduced as follows on Prim(A). Define for any $T \subset \mathrm{Prim}(A)$, $\overline{T} = h(k(T))$ then

(i) $\overline{\Phi} = \Phi$

(ii) $T \subset \overline{T} \quad \forall\, T \subset \mathrm{Prim}(A)$.

(iii) $\overline{\overline{T}} = \overline{T} \quad \forall\, T \subset \mathrm{Prim}(A)$.

(iv) $\overline{T_1 \cup T_2} = \overline{T}_1 \cup \overline{T}_2 \quad \forall\, T_1 , T_2 \subset \mathrm{Prim}(A)$.

Thus $T \to \overline{T}$ is a closure operation in Prim(A) and the unique topology defined by this closure operation is called the hull-kernel topology. Prim(A) with this hull-kernel topology is T_o. See C.E. Rickart [29].

Therefore a set $U \subset \text{Prim}(A)$ is open iff $\forall\, P \in \text{Prim}(A)$, $\exists\, b \in A$ such that $b \notin P$ but b belongs to every ideal contained in the complement of U.

1.1.17. Proposition. If A is an algebra with an identity then $\text{Prim}(A)$ is compact.

See C.E. Rickart [29].

1.1.18. Definition. $\text{Prim}(A)$ with the hull-kernel topology is called the structure space of A.

Some authors refer to this space as the spectrum, for example J. Dixmier and others, as the dual space, for example J.M.G. Fell.

Section 2. The Representation of a Real Commutative B*-Algebra.

1.2.1. Definition. Let A be an algebra over F (either the real or complex field). Then involution in A is a mapping $x \to x^*$ of A into itself such that $\forall\, x, y \in A$, $\lambda \in F$.

(i) $(x^*)^* = x$ (ii) $(x + y)^* = x^* + y^*$

(iii) $(\lambda x)^* = \bar{\lambda} x^*$ (iv) $(xy)^* = y^* x^*$.

The algebra is then called an involution algebra. If A is a normed algebra and $\| x^* \| = \| x \|$, $\forall\, x \in A$ then A is called a normed involution algebra.

1.2.2. Definition. A real (complex) B*-algebra A is a Banach involution algebra A over the reals (the complexes) such that

$\| x \|^2 = \| x^* x \| \quad \forall x \in A$.

$C_o(\Omega)$ will denote the space of continuous complex valued functions vanishing at infinity defined on Ω a locally compact Hausdorff space.

$C_o^R(\Omega)$ will denote the space of continuous real valued functions vanishing at infinity defined on Ω a locally compact Hausdorff space.

$B(H)$ will denote the space of bounded linear operators from a Hilbert space H into itself.

Examples of B*-Algebras.

(i) $C_o^R(\Omega)$ is a real B*-algebra with involution $f^* = f$, $\forall f \in C_o^R(\Omega)$.

(ii) $C_o(\Omega)$ is a complex B*-algebra with involution $f^* = \bar{f}$, $\forall f \in C_o(\Omega)$.

(iii) $B(H)$ with involution T^* the adjoint of $T \in B(H)$ is a B*-algebra.

1.2.3. Definition. A *C*-algebra* is a uniformly closed self-adjoint subalgebra of $B(H)$ over the reals or the complexes.

Every C*-algebra is a B*-algebra and in the complex case it can be shown that every B*-algebra is isometrically -*- isomorphic to a C*-algebra, hence complex B*-algebras are sometimes called abstract C*-algebras. We shall now show that under suitable conditions a real B*-algebra is isometrically -*- isomorphic to a real C*-algebra.

1.2.4. Definition. A an involution algebra, H a Hilbert space then a representation of A is a *- homomorphism T of A into $B(H)$. That is $T_{x^*} = (T_x)^*$, $\forall\ x \in A$.

The dimension of T denoted by dim T is the dimension of the Hilbert space H , and H is called the space of T and is denoted by H_T .

In the case that A is a complex B*-algebra, we have the equivalence of strict and topological irreducibility.

1.2.5. Theorem. A a complex B*-algebra then T a representation of A is strictly irreducible iff it is topologically irreducible.

See C.E. Rickart [29].

For a normed algebra A define $\upsilon(x) = \lim\limits_{n\to\infty} \| x^n \|^{\frac{1}{n}}$ and then it can be shown that if A is complete

$\upsilon(x) = \sup \{|\lambda| ; \lambda$ belongs to the spectrum of x in $A\}$

1.2.6. Definition. A a Banach algebra with involution, then the involution will be said to be hermitian if every hermitian element has a real spectrum and A will be called symmetric if $\forall\ x \in A$, $-x^*x$ is quasi-regular.

H_A will denote the set of hermitian elements of A. $Sp_A x$ will denote the spectrum of x in A.

1.2.7. Lemma. A a symmetric Banach algebra with involution and an identity such that the involution is continuous, then:

$\nu(x + h) \leqslant \nu(x) + \nu(h) \quad \forall\ x, h \in H_A$.

See C.F. Rickart [29].

1.2.8. Theorem. A a real B*-algebra such that $Sp_A x^*x \geqslant 0 \quad \forall\ x \in A$, is isometrically -*-isomorphic to a real C*-algebra over a real or complex Hilbert space.

Proof. The involution in A is hermitian because $Sp_A x^*x \geqslant 0$, $\forall\ x \in A$. Because $Sp_A x^*x \geqslant 0$ the algebra is symmetric and $\| x^* \| = \| x \|$, $\forall\ x \in A$. Hence by Lemma 1.2.7., $\nu(x + h) \leqslant \nu(x) + \nu(h) \quad \forall\ x, h \in H_A$ and $\nu(\lambda x) = \lambda\nu(x) \quad \forall\ \lambda \in \mathbb{R}$ and $x \in H_A$. Hence for a fixed element $a \in A$ by the Hahn-Banach Theorem $\exists$ a real linear functional say g on H_A such that $g(h) \leqslant \nu(h) \quad \forall\ h \in H_A$ and $g((aa^*)^2) = \nu((aa^*)^2)$. Decompose $x = h + k$ where $h = \frac{1}{2}(x + x^*) \in H_A$ and $k = \frac{1}{2}(x - x^*)$. Define $f(x) = g(h) \quad \forall\ x \in A$. Since $\nu(-x^*x) \leqslant 0$ we have $f(x^*x) \geqslant 0$. Thus f is a real positive linear functional such that $f(e) = 1$. Hence we can construct a representation Ψ of A onto a real Hilbert space H_Ψ . Moreover if $aa^* \neq 0$ then $\Psi(a) \neq 0$. Therefore the kernel of $\Psi = \{ a ; aa^* \neq 0 \}$. But $aa^* = 0 \Rightarrow a = 0$ hence the representation Ψ is faithful. Thus A is isometrically-*-isomorphic to a real C*-algebra over a real Hilbert space.

We observe that a real C*-algebra A over a real Hilbert space H is isometrically -*- isomorphic to a real C*-algebra A' over a complex Hilbert space H_c. Define $H_c = \{x + iy \; ; \; x, y \in H\}$. We shall now define addition, scalar multiplication and an inner product in H_c making it into a complex Hilbert space.

$$x + iy = u + iw \quad \text{iff } x = u \quad \text{and} \quad y = w .$$

Define $\forall \, x_i \, , \, y_i \in H \, , \, i = 1 \, , \, 2$ and $\forall \, \alpha \, , \, \beta \in \mathbb{R}$.

i) $(x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2)$.

ii) $(\alpha + i\beta)(x + iy) = \alpha x - \beta y + i(\alpha y + \beta x)$.

iii) $(x_1 + iy_1, x_2 + iy_2) = (x_1, x_2) + (y_1, y_2) + i((y_1, x_2) - (x_1, y_2))$.

$\forall \, a \in A$ define a mapping $a' : x + iy \to ax + iay$, $x, y \in H$. Then a' is a bounded linear operator over H_c with $\| a' \| = \| a \|$. Define $A' = \{a' \; ; \; a \in A\}$. Hence the mapping $a \to a'$ gives an isometric -*- isomorphism of A onto A'.

We shall now answer the question about what conditions are necessary to represent a real algebra A as the algebra $C^R(X)$ of all continuous real-valued functions defined on a compact Hausdorff space X . We shall need the Stone-Weierstrass theorem to show us when a subalgebra of $C^R(X)$ is in fact equal to $C^R(X)$.

1.2.9. Theorem. Let D be a subalgebra with identity of $C^R(X)$ which separates the points of X, (that is $\forall\, x, y \in X, \exists\, f \in D$ such that $f(x) \neq f(y)$) then D is dense in $C^R(X)$.

See C.E. Rickart [29].

1.2.10. Definition. A partially ordered vector space denoted by o.s. is a vector space V over the reals with a partial ordering given by a set of positive elements P, the so called positive cone of V. When $a - b \in P$, we write $a \geqslant b$. The following conditions are satisfied by P.

i) $a, b \in P \Rightarrow a + b \in P$, $\alpha \in \mathbb{R}^+ \Rightarrow \alpha a \in P$, $\forall\, a \in P$ and $P \cap -P = 0$.

ii) $\exists\, e \in V$ the order unit such that $\forall\, a \in V$, $\exists\, \alpha \in \mathbb{R}^+$ such that $-\alpha e \leqslant a \leqslant \alpha e$.

V is said to be archimedean if it satisfies in addition:

iii) $\alpha e \geqslant a \quad \forall\, \alpha \in \mathbb{R}^+ \Rightarrow 0 \geqslant a$, and is denoted by a.o.s.

Example. $C^R(X)$ the space of all continuous real valued functions defined on a compact Hausdorff space X is an a.o.s.

1.2.11. Definition. A linear subspace I of V with the property that $b \in I$ whenever $-a \leqslant b \leqslant a$ for some $a \in I$ is called an order ideal of V. Hence if $r \geqslant c \geqslant s$ with $r, s \in I$ then $c \in I$.

It can be shown that if V is an o.s. and I a maximal order ideal of V that V/I is a simple a.o.s. which is order isomorphic to $\mathbb{R}$.

We shall now define a norm on an a.o.s. V in the following way. $\forall\, a \in V$ define $\| a \| = \sup \{ |\alpha|, |\beta| \}$ where $\alpha = \inf \{ \alpha'; \alpha' e \geqslant a \}$ and $\beta = \sup \{ \beta'; \beta' e \leqslant a \}$ then V becomes a normed linear space and $\alpha e \geqslant a \geqslant \beta e$.

1.2.12. Definition. f a linear functional defined on an o.s. V is said to be positive if it is positive on all positive elements of V and normalized if $f(e) = 1$. A positive normalized linear functional will be abbreviated to p.n.l.f.

Hence to every maximal order ideal M of an o.s. V there corresponds a p.n.l.f. because V/M is isomorphic to $\mathbb{R}$ and the converse also holds. Let $P^* = \{ f;\ f$ is a p.n.l.f.$\}$, V^* the space of all bounded linear functionals of V. Then P^* is closed in the w*-topology on V^* hence compact, as it is contained in the unit sphere. Let $M(V)$ denote the set of maximal order ideals of V with the topology which makes the one to one correspondence with P^* a homeomorphism. An element $a \in V$ maps into a function $\bar{a}$ on $M(V)$ where $\bar{a}(M)$ is the image of a under the isomorphism of V/M with the reals.

1.2.13. Definition. An extreme ideal is a maximal order ideal which corresponds to an extreme point of the compact convex set P^*. Let $M(V)_e$ denote the set of extreme ideals.

1.2.14. Definition. An ordered algebra A is an algebra which in addition to being an o.s. satisfies the following conditions.

i) $\forall\ a \in A$, $ae = ea = a$ where e is the order unit.

ii) $a, b \geq 0$ then $ab \geq 0$.

So an ordered algebra necessarily has an identity. Now using the Stone-Weierstrass theorem we have the following representation theorem.

1.2.15. <u>Theorem</u>. A an archimedean ordered algebra which is complete in its norm is isometrically isomorphic to $C^R(X)$ the space of continuous real valued functions defined on the compact Hausdorff space $X = M(A)_e$.

See R.V. Kadison [16].

1.2.16. <u>Definition</u>. A normed algebra A is said to be <u>strictly</u> real if $-a^2$ has a quasi-inverse in $\bar{A}$ the completion of A, $\forall\ a \in A$ or in the case A has an identity e if $a^2 + e$ has an inverse in $\bar{A}$ the completion of A .

A strictly real Banach algebra A with identity such that $\| a^2 \| = \| a \|^2$ is necessarily commutative and using Theorem 1.2.15 we can represent A as a full algebra of continuous real valued functions.

1.2.17. <u>Theorem</u>. A a strictly real Banach algebra with identity such that $\| a^2 \| = \| a \|^2$, $\forall\ a \in A$ is isometrically isomorphic to $C^R(X)$ where X is the compact Hausdorff space of extreme ideals.

See R.V. Kadison [16].

1.2.18. <u>Corollary</u>. A a strictly real commutative B^*-algebra is isometrically isomorphic to $C^R(X)$.

<u>Proof</u>:

$$\| x^2 \|^2 = \| x^{*2}x^2 \| = \| (x^*x)^* x^*x \| = \| x^*x \|^2 = \| x \|^4$$

Hence: $\| x^2 \| = \| x \|^2 \quad \forall\, x \in A$ and therefore we can apply theorem 1.2.17.

We shall now consider the question of the representation of a real B^*-algebra over the set of all non-zero complex valued homomorphisms of the algebra called the carrier space.

Let A be a real or complex commutative algebra and denote by Φ_A the class of all non-zero homomorphisms of A into the field of complex numbers. Let φ_∞ denote the zero homomorphism and $\Phi_{A\infty}$ the class Φ_A plus φ_∞.

We then have the well known result that if A is a normed commutative algebra, M a closed maximal modular ideal, then A/M is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Hence if A is not a radical algebra it necessarily contains maximal modular ideals and if A is complete then every maximal modular ideal gives rise to an element of Φ_A. Hence Φ_A is non-void. The converse also holds because the kernel of φ where $\varphi \in \Phi_A$ is a maximal modular ideal.

$\forall\, x \in A$ define $\hat{x} : \Phi_A \to \mathbb{C}$ by $\hat{x}(\varphi) = \varphi(x)$.

We introduce the weakest topology into Φ_A for which each of the functions $\hat{x}$ is continuous. An arbitary neighbourhood of a point $\varphi \varepsilon \Phi_A$ is
$V_\varphi(x_1, ..x_n;\epsilon) = \{\varphi' ; | \hat{x}_i(\varphi) - \hat{x}_i(\varphi') | < \epsilon \}$ where $x_1,..x_n \varepsilon A$ and $\epsilon > 0$.

This topology is known as the Gelfand topology on $\Phi_{A\infty}$ and the mapping $x \to \hat{x}$ is called the Gelfand transformation of x.

1.2.19. Definition. The space Φ_A with the above defined topology is called the carrier space of the algebra A.

1.2.20. Proposition. A a commutative Banach algebra, then Φ_A with the Gelfand topology is a locally compact Hausdorff space and if A has an identity Φ_A is compact.

See C.E. Rickart [29].

Let τ be a homeomorphism of a compact Hausdorff space Ω onto itself. Denote by $C_o(\Omega,\tau) = \{ f \varepsilon C_o(\Omega) ; f(\omega^\tau) = \overline{f(\omega)}$ $\forall \omega \varepsilon \Omega \}$. Then $C_o(\Omega,\tau)$ is a real B*-algebra.

We shall now state a representation theorem for a real commutative B*-algebra as an algebra of not necessarily real valued functions whereas theorem 1.2.17. represented a real B*-algebra which satisfied certain conditions as an algebra of real valued functions.

1.2.21. Theorem. Let A be a real commutative B*-algebra, such that $\| x \| \leqslant \| x^*x + y^*y \| \quad \forall x, y \in A$ then $\exists$ an involutoric homeomorphism τ of Φ_A onto itself such that $x \to \hat{x}$ is an isometric - * - isomorphism of A onto the real B*-algebra $C_0(\Phi_A, \tau)$.

See C.E. Rickart [29].

Section 3. The Representation of a Complex Commutative B*-Algebra.

In the previous section it was noted that there was a correspondence between the carrier space Φ_A of a commutative Banach algebra A and $M(A)$ the space of maximal modular ideals. In the case when A is complex this correspondence is one to one but unfortunately when A is real this is not so.

When A is a complex commutative Banach algebra then the set Prim(A) equals the set $M(A)$ which equals the set Φ_A. We can define the hull-kernel topology on Φ_A in the following way.

1.3.1. Definition. Let B be a subset of A, F a subset of Φ_A then the set $h(B) = \{ \varphi \,;\, \varphi \in \Phi_A \text{ and } \hat{a}(\varphi) = 0 \ \forall a \in B \}$ is the hull of B in A and $k(F) = \{ a \,;\, a \in A \text{ and } \hat{a}(\varphi) = 0 \ \forall \varphi \in F \}$ is the kernel of F in A.

By defining a closure operation as follows $\overline{F} = h(k(F))$ we have the hull-kernel topology on Φ_A which makes the correspondence with Prim(A) a homeomorphism.

We shall now establish what conditions are necessary for the hull kernel topology to be equivalent to the Gelfand topology on Φ_A.

1.3.2. Definition. A a complex commutative Banach algebra is completely regular if for every closed set F in Φ_A and point $\varphi_0 \notin F$, $\exists\ a \in A$ such that $\hat{a}(\varphi_0) \neq 0$ while $\hat{a}(F) = 0$.

1.3.3. Theorem. A a complex commutative Banach algebra then the hull kernel topology in Φ_A is equivalent to the Gelfand topology in Φ_A iff A is completely regular.

See C.E. Rickart [29].

Hence as a complex commutative B*-algebra is completely regular Prim(A) and Φ_A are homeomorphic.

Example. $C(X)$ the space of all continuous complex valued functions defined on a compact Hausdorff space X has its set of maximal modular ideals equal to X. That is $M(C(X)) = X$, therefore $\text{Prim}(C(X)) = X$.

1.3.4. Definition. $C_0(\Omega)$ the space of continuous complex valued functions which vanish at infinity defined on a locally compact Hausdorff space Ω is said to strongly separate the points of Ω if for any $\omega_1 \neq \omega_2 \in \Omega$, $\exists\ f \in C_0(\Omega)$ such that $f(\omega_1) \neq f(\omega_2)$ and $\forall\ \omega \in \Omega$, $\exists\ f \in C_0(\Omega)$ such that $f(\omega) \neq 0$.

We shall now state the Stone-Weierstrass theorem which will be needed to prove the representation theorem for the complex algebra.

1.3.5. <u>The Stone-Weierstrass Theorem</u>. Let D be a subalgebra of $C_o(\Omega)$ which strongly separates the points of Ω and is self-adjoint then D is dense in $C_o(\Omega)$.

See C.E. Rickart [29].

We now have the Gelfand Naimark representation theorem which is proved using the above Stone-Weierstrass theorem.

1.3.6. <u>Gelfand Naimark Theorem</u>. The Gelfand transformation of a complex commutative B^*-algebra A is an isometric - * - isomorphism of A onto $C_o(\Phi_A)$ where Φ_A is the carrier space of A.

See C.E. Rickart [29].

In the case that A is a complex commutative C^*-algebra with an identity then the Gelfand transformation is an isometric - * - isomorphism of A with $C(\Phi_A)$. Hence we have represented A as the algebra of continuous complex valued functions defined on its structure space, as Φ_A is homeomorphic to Prim(A) .

1.3.7. <u>Definition</u>. A a complex involution algebra. A linear complex valued functional f defined on A is said to be <u>positive</u> if $f(x^*x) \geq 0 \quad \forall\, x \in A$. If A is normed then a <u>state</u> of A is a positive functional f such that $\| f \| = 1$. $S(A)$ will denote the set of states of A.

1.3.8. Definition. A a complex B*-algebra. An hermitian element $x \in A$ is said to be positive if the spectrum of x is positive. A^+ will denote the set of positive elements.

A^+ is closed convex cone such that $A^+ \cap -A^+ = 0$ and $x \in A^+$ iff $x = yy^*$ with $y \in A$.

1.3.9. Proposition. A a complex B*-algebra with an identity, f a continuous linear functional on A is positive iff $\| f \| = f(e)$.

See J. Dixmier [6].

1.3.10. Definition. A a normed complex involution algebra then f a positive functional on A is said to be pure if $f \neq 0$ and every positive functional on A dominated by f is of the form λf with $0 \leqslant \lambda \leqslant 1$.

$P(A)$ will denote the set of pure states.

There exists a correspondence between the positive functionals on a complex Banach involution algebra and the representations of this algebra as this next proposition shows us. $\hat{A}$ denotes the equivalence classes of irreducible representations of A.

1.3.11. Proposition. A a complex Banach involution algebra with an approximate identity:

i) f a continuous positive functional on A then $\exists$ a representation T of A on H_T such that $f(x) = (T_x\xi, \xi)$ where $\xi \in H_T$, $\forall x \in A$.

ii) T a representation of A on H_T and $\xi \varepsilon H_T$ then $\forall\, x \,\varepsilon\, A$, $f(x) = (T_x\xi,\xi)$ is positive continuous functional on A and the representation T' defined by f according to (i) is equivalent to T.

iii) f a continuous positive functional on A and T the representation of A defined by (i) is topologically irreducible iff f is pure then $\| f \| = (\xi,\xi)$ where $f(x) = (T_x\xi,\xi)$.

iv) This natural mapping of $P(A) \to \hat{A}$ is continuous and open if A is a complex B^*-algebra.

See J. Dixmier [6].

Hence we have a natural mapping $P(A) \to \hat{A}$ and the question as to when this mapping is one to one, is answered by the next proposition.

1.3.12. Proposition. A an involution algebra, T a non-zero topologically irreducible representation of A, $\xi_1,\xi_2 \,\varepsilon\, H_T$, f_1, f_2 the positive functionals defined by T as follows $f_1(x) = (T_x\xi_1, \xi_1)$, $f_2(x) = (T_x\xi_2, \xi_2)$ then $f_1 = f_2$ iff $\exists\, \lambda \,\varepsilon\, ¢$ such that $| \lambda | = 1$ and $\xi_2 = \lambda\xi_1$.

See J. Dixmier [6].

Hence the mapping of $P(A)$ onto $\hat{A}$ is bijective iff every topologically irreducible representation of A is of dimension one. Therefore if A is a commutative complex B^*-algebra $\hat{A}$ is homeomorphic with $P(A)$ because every irreducible representation of A is of dimension one. In the case that A has an identity $P(A)$ is the set of extreme points of $S(A)$ and one could have defined a pure state as being an extreme point of $S(A)$.

The Stone-Weierstrass theorem could therefore be restated as follows, " A a commutative complex B*-algebra, B a sub B*-algebra which separates $P(A)$ then $B = A$," because $P(A)$ coincides with Φ_A.

In the next chapter the type of Stone-Weierstrass theorem used in the representation theorem is the above generalized by J. Glimm, [11], to the non-commutative case.

Notes and Remarks.

Theorem 1.2.8. is a recent result due to T. Ono [28]. It has been proved by L. Ingelstam [14] in 1964, whose proof was based on a complexification of a real Banach involution algebra.

Theorem 1.2.21. is due to R. Arens and I. Kaplansky [1] and has been generalized by L. Ingelstam [14] to the case where the algebra has no identity. Namely, " A a strictly real Banach algebra such that $\| x^2 \| \geq \alpha \| x \|^2$ for all $x \in A$ and some fixed $\alpha > 0$ then A is isomorphic and homeomorphic to $C^R(\Phi_A)$ ".

The classical result of Theorem 1.3.6. is due to I.M. Gelfand and M.A. Naimark [10].

CHAPTER II.

REPRESENTATION OF BANACH ALGEBRAS BY FUNCTIONS CONTINUOUS IN THE NORM.

In this chapter a Banach algebra will be an algebra over the complexes. Every complex B*-algebra is isometrically - * - isomorphic to a complex C*-algebra, so that the term C*-algebra can be taken to mean either a complex B*-algebra or a complex C*-algebra.

We shall try to represent a non-commutative C*-algebra as a C*-algebra of vector valued functions, that is elements of $\prod_{t \in T} A_t$ where T is the structure space and the A_t are C*-algebras. As the set $\{A_t ; t \in T\}$ has no topology on it, we shall consider the vector valued functions which are continuous in the norm. T the structure space will either be the space of primitive ideals of the algebra or the space of equivalence classes of irreducible representations, which unfortunately, in general is not Hausdorff even though it is always locally compact.

In Section 1 certain properties of the structure space of a C*-algebra will be proved and the important result in this section is Theorem 2.1.12. which shows us that the norm continuity of the vector valued functions is related to the structure space being

Hausdorff. In Section 2, G.C.R. and C.C.R. algebras will be defined and our main result is Theorem 2.2.17. which shows us that every C.C.R. algebra has a composition series whose factors have Hausdorff structure spaces.

In Section 3, in the setting of a continuous field of Banach spaces a suitable type of non-commutative Stone-Weierstrass theorem, (Corollary 2.3.16.) will be proved which will enable us to prove the representation theorem for G.C.R. algebras. (Theorem 2.3.19.)

Section 1. Properties of the Structure Space of an Algebra.

We shall start this section with some well known properties of C*-algebras.

2.1.1. Proposition. A a C*-algebra then:

i) If x is an hermitian element belonging to A, then $Sp_A x \in \mathbb{R}$.

ii) I a closed ideal in A then A/I is a C*-algebra.

iii) B a sub C*-algebra of A, $x \in B$ then $Sp_B x = Sp_A x$.

See J.Dixmier [6].

A a C*-algebra, x an hermitian element belonging to A then B the closed subalgebra generated by x consists of all the continuous complex valued functions vanishing at infinity defined on a locally compact Hausdorff space X. (X is homeomorphic to the non-zero spectrum of x). Let p be any continuous real function with $p(0) = 0$. Then $p(x)$ is a well defined hermitian element belonging to B. Suppose further that I is a closed ideal in A and let $x(I)$ be the homomorphic image of x mod I. Then $p(x(I))$ may be formed and in fact $p(x(I)) = p(x)(I)$. This is clear if p is a polynomial and as the general p is a uniform limit of such polynomials and the map from $A \to A/I$ is continuous, the result follows.

If F is a set of closed ideals with zero intersection and x an hermitian element of A such that $x(s) \geq 0$, $\forall s \in F$, then $x \geq 0$. To see this, define p by $p(t) = 0$ for $t \geq 0$, $p(t) = -t$ for $t \leq 0$. Then $p(x(s)) = p(x)(s) = 0$. Hence $p(x) = 0 \Rightarrow x \geq 0$.

This next proposition gives us a type of continuity of the spectrum.

2.1.2. Proposition. Let X be a topological space at each point of which a C*-algebra A_x is given and let A be a self-adjoint algebra of functions f on X with $f(x) \in A_x$ $\forall x \in X$ such that:

i) $\| f(x) \|$ is bounded and A is complete under the norm $\| f \| = \sup_{x \in X} \| f(x) \|$.

ii) If $f(x) = 0$ then $\forall \epsilon > 0$, $\exists U_x$ a neighbourhood of x such that $\forall y \in U_x$, $\| f(y) \| < \epsilon$.

Then $\forall f \in A$ such that $f = f^*$ we have $\forall x \in X$, $\forall \epsilon > 0$,

$\exists\ U_x$ a neighbourhood of x such that $\forall\ y \in U_x$, $Sp_{A_y} f(y)$ is contained in an ϵ - neighbourhood of the set consisting of zero and the spectrum of $f(x)$.

See M.A. Naimark [27].

2.1.3. Definition. Let $(A_i)\ i \in I$ be a family of C*-algebras. Let A denote the set of $(x_i)\ i \in I$ such that $x_i \in A_i$, $\forall\ i \in I$ and $\sup_{i\in I} \| x_i \| < \infty$ then under the operations:

$$(x_i) + (y_i) = (x_i + y_i) \qquad \lambda(x_i) = (\lambda x_i)$$
$$(x_i)(y_i) = (x_i y_i) \qquad (x_i)^* = (x_i^*)$$

and norm $\| (x_i) \| = \sup_{i\in I} \| x_i \|$ we see that A is a C*-algebra called the C*-sum of the A_i.

2.1.4. Proposition. I, J closed ideals in a C*-algebra A such that $I + J$ is dense in A and $I \cap J = \{0\}$. Then $I + J = A$ and A is the C*- sum of I and J.

Proof: There is a natural map of A into the C*- sum of A/I and A/J. Because $I + J$ is dense in A, the image of A is dense in the C* sum of A/I and A/J. Hence the mapping is onto and is an isometry.

For A an algebra Prim(A) denotes the set of primitive ideals and $\hat{A}$ the set of equivalence classes of irreducible representations. There is a natural mapping from $\hat{A}$ onto Prim(A) defined by $T \to$ kernel of T, $\forall\ T \in \hat{A}$. $\hat{A}$ is given the weakest topology so that the mapping from $\hat{A} \to \text{Prim}(A)$ (where $\text{Prim}(A)$ has the hull kernel topology) is continuous.

J.M.G. Fell in [8], defines the topology in $\hat{A}$ in the following way. For $W \subset \hat{A}$ define the closure of W denoted by $\overline{W} = \{ T \in \hat{A} \; ; \bigcap_{S \in W} k(S) \subset k(T) \}$, where $k(T)$ is the kernel of T. This topology on $\hat{A}$ is equivalent to the topology on $\hat{A}$ induced by $\text{Prim}(A)$.

Now if A has an identity then $\text{Prim}(A)$ is compact by Proposition 1.1.17. hence $\hat{A}$ is compact.

2.1.5. Proposition. The following conditions are equivalent for an algebra A:

i. $\hat{A}$ is T_0.

ii. Two irreducible representations of A with the same kernel are equivalent.

iii. The mapping $\hat{A} \to \text{Prim}(A)$ is a homeomorphism.

Proof:

ii $\Rightarrow$ iii Follows from the definitions.

iii $\Rightarrow$ i Because $\text{Prim}(A)$ is T_0.

i $\Rightarrow$ ii Let T^1, T^2 be two representations with the same kernel belonging to $\hat{A}$. Every open subset of $\hat{A}$ containing T^1 also contains T^2. Hence if $\hat{A}$ is T_0, $T^1 = T^2$.

Let A be a C*-algebra. We may represent an arbitrary element $a \in A$ as a function defined on Prim(A) by $a: P \to a(P)$ where $a(P)$ denotes the homomorphic image of a mod P. This functional representation preserves norm. That is

$\| a \| = \sup_{P \in \mathrm{Prim}(A)} \| a(P) \|$. In fact $\exists\, P \in \mathrm{Prim}(A)$ such that $\| a \| = \| a(P) \|$.

2.1.6. Lemma. Let A be an involution algebra with an identity. x an hermetian element belonging to A. Then $\lambda \in Sp_A x$ iff $\exists\, P \in \mathrm{Prim}(A)$ such that $\lambda \in Sp_{A/P}\, x(P)$.

Proof: If $x - \lambda$ is regular in A then $\bar{x} - \lambda$ is regular in A/P, $\forall\, P \in \mathrm{Prim}(A)$ where $\bar{x}$ denotes the image of x in A/P.

Suppose $x - \lambda$ is singular. Then $x - \lambda$ has no left inverse, therefore is contained in a maximal left ideal L. The kernel I of the natural representation of A on A/L is a primitive ideal of A and the image of $x - \lambda$ in A/I has no inverse.

2.1.7. Proposition. Let A be a C*-algebra. $\forall\, x \in A$ the function $P \to \| x(P) \|$ on Prim(A) attains its upper bound $\| x \|$.

Proof: Replacing x by x^*x, one reduces to the case where $x \in A^+$. One may also suppose that A has an identity. Then $\| x \| \in Sp_A x$, hence by the previous Lemma $\exists\, P \in \mathrm{Prim}(A)$ such that $\| x \| \in Sp_{A/P}\, x(P)$. Hence $\| x \| \leqslant \| x(P) \| \leqslant \| x \|$

2.1.8. <u>Proposition.</u> τ an ideal of a ring R. $Q = \{ P \in \text{Prim}(A); P \supset \tau \}$ then the mapping $P \to P/\tau$ is a homeomorphism of Q onto $\text{Prim}(R/\tau)$.

See N. Jacobson [15].

2.1.9. <u>Proposition.</u> Let A be any ring and B either:

i. a two-sided ideal in A.

ii. a subring of the form eAe where e is a non-zero idempotent.

iii. or a subring of the form $(1 - e)A(1 - e)$ where e is a non-zero idempotent.

Then there is a one to one correspondence between $\text{Prim}(B)$ and $Q = \{ P \in \text{Prim}(A) ; P \not\supset B \}$ given by $P \to P \cap B$. In fact this mapping is a homeomorphism.

See N. Jacobson [15].

If R is a primitive ring, then any ideal contained in R is again primitive and if e is a non-zero idempotent then eRe is again primitive.

2.1.10. <u>Lemma.</u> Let $a \in A$ be an hermitian element where A is a C*-algebra. Let E be a closed set of real numbers containing zero. Then $Z = \{ P ; P \in \text{Prim}(A)$ and $\text{Sp}_{A/P}a(P) \subset E \}$ is closed in $\text{Prim}(A)$.

<u>Proof:</u> If $\exists P \in \bar{Z}$ such that $\alpha \in \text{Sp}_{A/P}a(P)$ and $\alpha \notin E$ then let p be a continuous real valued function vanishing on E

but not on α. $p(a(P)) = (pa)(P)$. Then $p(a)$ vanishes on Z but not on $\overline{Z}$ contradicting the definition of the hull-kernel topology on Prim(A).

This next proposition shows us that A can be considered to be a set of functions vanishing at infinity.

2.1.11. Proposition. Let $a \in A$ a C*-algebra, $\epsilon > 0$. Then $K = \{ P \in \text{Prim}(A) \; ; \; \| a(P) \| \geq \epsilon \}$ is compact.

Proof: Because $\| a^*a \| = \| a \|^2$ we need only consider the case where $a = a^*$. Let $\{ F_j \; ; \; j \in J \}$ be a family of relatively closed subsets of K having void intersection. Let I_j be the intersection of the primitive ideals comprising F_j, H_0 the ideal generated by the I_j and $H = \overline{H}_0$. H is not contained in any of the primitive ideals $P \in K$. For if $H \subset R$, $R \in K$, then R contains each I_j. Hence by the definition of the topology on Prim(A) $R \in F_j$, $\forall \, j \in J$. This contradicts $\bigcap_{j \in J} F_j$ being void. A/H is a C*-algebra hence is semi-simple. Therefore by Proposition 2.1.8. H is the intersection of the primitive ideals containing it. Therefore $H = h(k(H))$. Let $L = h(H)$ and as $L \cap K$ is void $\forall \, Q \in L$, $\| a(Q) \| < \epsilon$.

Let $r = \sup_{Q \in L} \| a(Q) \|$. Let $a_1 = a(H)$. Then $r = \sup_{P \in \text{Prim}(A/H)} \| a_1(P) \|$. This sup is attained by Proposition 2.1.7. Hence $\exists \, Q_0 \in L$ such that $r = \| a(Q_0) \| < \epsilon$.

Let $p(t)$ be a continuous real valued function of the real variable t such that $p(t) = 0$ for $|t| \leqslant r$, $p(t) = 2$ for $|t| \geqslant \epsilon$ and p is linear between. Let $b = p(a)$. Then $b \in H$ since it vanishes on L and $\| b(P) \| = 2$ for P in K. Since H_o is dense in H, $\exists\ c \in H_o$ with $\| c - b \| < 1$. H_o is the smallest ideal generated by the I_j therefore $\exists\ I_1, .. I_n$ such that $c = x_1 + .. + x_n$ with $x_i \in I_i$, $\forall\ i = 1, .. n$. Then $F_1 \cap .. \cap F_n = \varphi$ because if $\exists\ P \in F_1 \cap .. \cap F_n$ then $P \supset I_i$ $\forall\ i = 1, .. n$. Hence $c \in P \Rightarrow c(P) = 0$ but $\| c(P) \| \geqslant 1$ throughout K.

The following theorem tells us that if Prim(A) is Hausdorff then the functional representation of A on Prim(A) is continuous in the norm.

2.1.12 Theorem. Prim(A) is Hausdorff iff $\forall\ a \in A$, $\| a(P) \|$ is continuous on Prim(A).

Proof: For any $Q, R \in$ Prim(A) $Q \neq R$, $\exists\ a \in A$ such that $a(Q) = 0$, $a(R) \neq 0$. The continuous real valued function $\| a(P) \|$ then yields disjoint neighbourhoods of Q, R. Let Prim(A) be Hausdorff. We shall show:

i. $a(P) = 0$ then $\forall\ \epsilon > 0\ \exists\ U_P$ a neighbourhood of P such that $\forall\ R \in U_P$, $\| a(R) \| < \epsilon$.

Take a fixed $Q \in \mathrm{Prim}(A)$. Let $J = \{ a \in A ; a(P)$ vanishes on some neighbourhood of $Q \}$. Put $I = \bar{J}$. I is a closed ideal in A , in fact $I = Q$. Since if $I \neq Q \Rightarrow \exists\, Q_o \in \mathrm{Prim}(A)$ such that $I \subset Q_o$ where $Q \neq Q_o$. By the Hausdorff property $\exists\, U_Q$ a neighbourhood of Q such that $Q_o \not\subset \bar{U}_Q$. But by the definition of the topology on Prim(A) , A contains an element vanishing on U_Q but not on Q_o . Hence we have arrived at a contradiction because $a(u) = 0 \Rightarrow a \in I \subset Q_o \Rightarrow a(Q_o) = C$.
Therefore we have proved (i).

We shall now pass to the general proof of continuity. Because $\| a^*a \| = \| a \|^2$ it suffices to consider $a \in A$ such that $a = a^*$. Let $Q \in \mathrm{Prim}(A)$ and $\epsilon > 0$ be given. Put $r = \| a(Q) \|$. It follows from Proposition 2.1.2. that in a suitable neighbourhood of Q , $\| a(P) \| < r + \epsilon$. To complete the proof it suffices to show that
$\{ P \in \mathrm{Prim}(A) ; \| a(P) \| > r - \epsilon \}$ is open. That is
$\{ P ; \| a(P) \| \leqslant r - \epsilon \}$ is closed. This follows from Lemma 2.1.10.

J. Dixmier [6] shows that for any C*-algebra A, $\hat{A}$ is locally compact hence Prim(A) is also locally compact.

We shall now try and find what conditions are necessary for Prim(A) to be Hausdorff.

2.1.13. <u>Theorem</u>. A a C*-algebra such that $\forall\, P \in \mathrm{Prim}(A)$, A/P is finite dimensional of dimension independent of P then Prim(A) is Hausdorff.

Proof: As A/P is finite dimensional $\forall\, P \in \text{Prim}(A)$ $\exists\, M$ a finite dimensional C*-algebra to which each A/P is isomorphic. Let Y be the space of all *-homomorphisms of A into M including the zero homomorphism, then the elements of Y can be considered to be the set of all the representations of A into a simple dual algebra. In the weak topology Y is a compact Hausdorff space and the elements of A are represented by continuous functions from Y to M. The orbit of a point $y \in Y$ is denoted by $O_y = \{ g \circ y \,;\, g \in G \}$, G being the group of *-automorphisms of M.

$\forall\, y \in Y$, $O_y \subset Y$ and $\bigcup_{y \in Y} O_y = Y$. Hence each primitive ideal gives rise to an orbit of points in Y. Now G is compact in its natural topology and the mapping from G onto an orbit is continuous. Hence the orbits are closed and we may form a well-defined quotient space X relative to this decomposition of Y. The points of X being in one to one correspondence with the set consisting of Prim(A) and a point at infinity. Being a continuous image of Y, X is again compact. We can no longer speak of elements of A as being represented by continuous functions $a(x)$ on X, but the function $\| a(y) \|$ is constant on orbits and so gives us a uniquely defined function on X which is again continuous. These functions $\| a(x) \|$ exist in sufficient abundance to separate the points of X, for given two distinct points of X we can find an $a \in A$ vanishing at one but not the other. Hence X is Hausdorff. As A contains all the real continuous functions that vanish at the zero homomorphism, X with 0 deleted is homeomorphic to Prim(A).

Section 2. G. C. R. and C. C. R. Algebras.

2.2.1 Definition. An operator A in a complete normed space X is said to be completely continuous if it maps every bounded set into a relatively compact set. $LC(X)$ will denote the set of completely continuous operators.

$LC(X)$ is a closed ideal in $B(X)$ and if instead of X we consider a Hilbert space H then $A \in LC(H) \Rightarrow A^* \in LC(H)$. Hence $LC(H)$ is a C*-algebra.

2.2.2. Proposition. Every simple dual C*-algebra is isometrically -*- isomorphic to the algebra of all completely continuous operators on some Hilbert space H.

See M.A. Naimark [27].

2.2.3. Definition. A C*-algebra is said to be elementary if $\exists$ a Hilbert space H such that A is isometrically -*- isomorphic to $LC(H)$.

2.2.4. Definition. A C*-algebra A is said to be C.C.R. if for every irreducible representation T of A and $\forall\, x \in A$, $T_x \in LC(H_T)$.

For T a non-zero irreducible representation of A a C*-algebra and $T(A) \subset LC(H_T)$ then $T(A) = LC(H_T)$. Hence if A is C.C.R., $\forall\, P \in \mathrm{Prim}(A)$, A/P is elementary.

2.2.5. Definition. A C*-algebra A is said to be G.C.R. if every non-zero C*-algebra quotient of A possesses a closed non-zero C.C.R. ideal.

2.2.6. Proposition. If A is C.C.R. (G.C.R.) then every sub-C*- algebra of A and every C*-algebra quotient of A is C.C.R. (G.C.R.) .

See J. Dixmier [6].

2.2.7. Proposition. A a C*-algebra, T an irreducible representation of A such that $T(A) \cap LC(H_T) \neq 0$ then every irreducible representation of A with the same kernel is equivalent to T.

See J. Dixmier [6].

So that in the case A is C.C.R. by Proposition 2.1.5. $\hat{A}$ and $\mathrm{Prim}(A)$ are homeomorphic.

2.2.8. Definition. A C*-algebra A is said to be N.G.C.R. if it does not possess any closed non-zero C.C.R. ideal.

Hence a N.G.C.R. algebra is one which does not possess any closed two-sided G.C.R. ideal.

2.2.9. Example. A commutative C*-algebra is C.C.R. because all its irreducible representations are one dimensional.

2.2.10. Definition. A composition series in a C*-algebra A is a well ordered ascending series of closed ideals $(I_\rho)_{0\leqslant\rho\leqslant\alpha}$ indexed by the ordinals such that $I_0 = 0$ and $I_\alpha = A$ and if $\rho \leqslant \alpha$ is a limit ordinal, then I_ρ equals the closure of $\bigcup_{\rho'<\rho} I_{\rho'}$.

We then have the following characterisation for a G.C.R. algebra.

2.2.11. Proposition. A is a G.C.R. algebra iff $\exists$ a composition series $(I_\rho)_{0\leqslant\rho\leqslant\alpha}$ such that $I_{\rho+1}/I_\rho$ is C.C.R.

See J. Dixmier [6].

In order to be able to represent A as a full algebra of of functions, we need its structure space to be Hausdorff. These next few theorems will enable us to prove our main theorem in this section that every C.C.R. algebra has a composition series such that $\mathrm{Prim}(I_{\rho+1}/I_\rho)$ is Hausdorff.

2.2.12. Lemma. Let $T_1, \ldots, T_n$ be self-adjoint completely continuous operators on a Hilbert space H, r a non-negative real number and α a vector of unit length in H such that $(I + T_1)^2 + \ldots + (I + T_n)^2 - r^2 I$ annihilates α, I being the identity operator. Then for each i there exists a self-adjoint completely continuous operator U_i such that $\alpha(I + U_i) = 0$

and $\| U_i - T_i \| \leq 3r$.

Proof: Let E be the projection on α, and define $V_i = E(I + T_i) + (I + T_i)E - E(I + T_i)E$. Then V_i is self-adjoint and completely continuous. We have

$$\| V_i \| \leq 3\| E(I + T_i) \| = 3\| \alpha(I + T_i) \| ,$$
$$\| V_i \|^2 \leq 9 \Sigma(\alpha(I + T_i)^2, \alpha) = 9r^2,$$

and so $\| V_i \| \leq 3r$. We note that $\alpha V_i = \alpha(I + T_i)$. Hence the choice $U_i = T_i - V_i$ satisfies the requirements of the lemma.

2.2.13. Theorem. The structure space X of a C.C.R-algebra A is of the second category.

Proof: Suppose on the contrary that X is the union of an increasing sequence C_n of closed nowhere dense sets. We proceed to define stepwise an array of objects.

a) An increasing sequence of integers $k(1), k(2), \ldots$,
b) For $j = 2, 3, \ldots$, a point (equal to a primitive ideal) P_j; we realize the algebra A/P_j as the algebra of all completely continuous operators on a Hilbert space H_j,
c) A vector α_j of unit norm in H_j,
d) Self-adjoint elements $a(i, j) \in A$, defined for $j = 2, 3, \ldots$ and $1 \leq i \leq j$.

To begin the process, we set $k(1) = 1$, $k(2) = 2$ and take $a(1, 2)$ to be a self-adjoint element vanishing on C_1 and having -1 in its spectrum at a certain point P_2 not in C_2;

and we pick α_2 to be a vector of unit norm annihilated by the homomorphic image of $1 + a(1, 2)$ mod P_2. (This formal use of 1 is legal in all the ensuing manipulations even if A lacks a unit element.) Then take $a(2, 2)$ to be a self-adjoint element vanishing on C_2 and such that α_2 is also annihilated by $1 + a(2, 2)$ at P_2.

Now suppose the selection of $k(j)$, P_j, α_j, $a(i,j)$ has been made for $i \leqslant j \leqslant n$ so as to satisfy

1) P_{j-1} is not in $C_{k(j-1)}$ but is in $C_{k(j)}$,
2) α_j is annihilated by the homomorphic image of $1 + a(i,j)$ mod P_j,
3) $a(i, j)$ vanishes on $C_{k(i)}$,
4) $a(i, j - 1)$ and $a(i, j)$ coincide on $C_{k(j)}$,
5) $\| a(i, j - 1) - a(i, j) \| \leqslant 2^{-j}$.

Our selections in the preceding paragraph satisfied these assumptions, insofar as they were as yet meaningful. We show how to push on to $n + 1$. Take $k(n + 1)$ large enough so that P_n is in $C_{k(n+1)}$. We write

$$b_n = [1 + a(1, n)]^2 + \dots + [1 + a(n, n)]^2 - n,$$

observing that b_n is an actual ring element. It follows from (2) that b_n has $-n$ in its spectrum at P_n. We claim that there exists a point P_{n+1} outside $C_{k(n+1)}$, where the spectrum of b_n contains a number arbitrarily close to $- n$; concretely, we specify that the spectrum of $b_n(P_{n+1})$ shall contain a number lying within $m = 2^{-2n-8}$ of $-n$. For if not, let $p(t)$ be a continuous real-valued function which is non-zero at $-n$ and vanishes outside the open interval from $-n-m$ to $-n+m$. Then $p(b_n)$ vanishes in the complement of $C_{k(n+1)}$ but is different from 0 at P_n, and this contradicts the assumption that $C_{k(n+1)}$ is nowhere dense (whence its complement is dense). Hence we may choose $P_{n+1} \notin C_{k(n+1)}$ in such a way that $b_n(P_{n+1})$ has in its spectrum a real number s with $|s+n| < m$. In accordance with the above notation we realize A/P_{n+1} as the algebra of all completely continuous operators on a Hilbert space H_{n+1}. We note that s is non-zero, and consequently we may find in H_{n+1} a unit vector α_{n+1} which is a characteristic vector for the operator $b_n(P_{n+1})$, corresponding to the characteristic root s. Let us write T_i for the operator representing $a(i, n)(P_{n+1})$. Then, in view of the definition of b_n, we find that the operator

$$(I + T_1)^2 + \ldots + (I + T_n)^2 - (s + n)I$$

annihilates α_{n+1}. We are now in a position to apply Lemma 2.2.12. and we deduce the existence of self-adjoint completely continuous operators $U_1, \ldots, U_n$ on H_{n+1} with $\alpha_{n+1}(I + U_i) = 0$ and $\| T_i - U_i \| \leq 3m^{\frac{1}{2}} < 2^{-n-2}$.

Let us write J for the intersection of the ideals comprising $C_{k(n+1)}$, and $K = J \cap P_{n+1}$. We note that J is not contained in P_{n+1}, for $C_{k(n+1)}$ is a closed set not containing P_{n+1}. Since there are no closed ideals between P_{n+1} and A, $J + P_{n+1}$ is dense in A. Hence [Lemma 2.1.4.], A/K is the direct sum of A/J and A/P_{n+1}, in the C* as well as in the algebraic sense. It is therefore possible to find in A/K a (unique) element g_i which vanishes mod J, and maps on $U_i - T_i$ mod P_{n+1}. We shall have $\| g_i \| = \| U_i - T_i \| < 2^{-n-2}$. Going back to A, we can find a self-adjoint element h_i, mapping on g_i mod K, and with norm arbitrarily close to $\| g_i \|$; we do this so as to get $\| h_i \| < 2^{-n-1}$.

We are ready to define $a(i, n + 1)$ as $a(i, n) + h_i$. In view of the estimate for $\| h_i \|$, this satisfies condition (5) above. If we further note that $a(i, n + 1)$ was constructed so as to agree with $a(i, n)$ on $C_{k(n+1)}$, and map on U_i at P_{n+1}, we verify conditions (2) - (4). P_{n+1} was selected so as to verify (1).

To complete this step of the induction, it now only remains to choose $a(n + 1, n + 1)$. For this purpose we need only choose it to be a self-adjoint element vanishing on $C_{k(n+1)}$, and such that its image mod P_{n+1} annihilates a_{n+1}. That such a selection is possible follows from the fact that (in the notation above) A/K is the direct sum of A/J and A/P_{n+1}.

Having completed the construction, we proceed as follows.

From (5) we have that for each fixed i, the sequence $a(i, j)$ converges, say to c_i. We note two properties of the c's. Since property (3) is evidently preserved on passage to the limit, we have:

(6) c_i vanishes on $C_{k(i)}$.

Next we deduce from (4) and the fact that the C's are an increasing sequence of sets, that $a(i, j - 1)$ coincides with $a(i, s)$ on $C_{k(j)}$ for all $s \geq j$. Now $P_{j-1} \varepsilon C_{k(j)}$ by (1) ; it then follows from (2) that a_{j-1} is annihilated by the homomorphic image at P_{j-1} of every $a(i, s)$ with $s \geq j$. This is preserved in the limit, and we state:

(7) a_j is annihilated by the homomorphic image of $1 + c_i$ mod P_j.

We are now ready to exhibit the contradiction. Consider the right ideal defined by

$$I = (1 + c_1)A + .. + (1 + c_n)A + .. \ .$$

Let Q be any primitive ideal in A, say $Q \varepsilon C_{k(n)}$. Since, by (6) c_n vanishes on $C_{k(n)}$, we have $c_n \varepsilon Q$. The equation $x = (1 + c_n)x - c_n x$ shows that $I + Q = A$. Next, I is regular, since for example $- c_1$ is a left unit for I. If $I \neq A$, then I can be embedded in a regular maximal right ideal M. There is a primitive ideal Q contained in M (Q is the kernel of the natural representation of A on A/M) , and this contradicts $I + Q = A$. Hence $I = A$, and for suitable elements d_i :

$$(1 + c_1)d_1 + \ldots + (1 + c_n)d_n = - c_1$$

or

$$(1 + c_1)(1 + d_1) + (1 + c_2)d_2 + \ldots + (1 + c_n)d_n = 1$$

This contradicts the assertion in (7) that the homomorphic images mod P_n of $1 + c_1, \ldots, 1 + c_n$ all annihilate a_n . This completes the proof.

I. Kaplansky [17] has the following definition.

2.2.14. <u>Definition.</u> F a field, $F[x_1, \ldots x_n]$ denotes the free algebra generated by n indeterminates over F . A an algebra over F is said to satisfy a polynomial identity if
$\exists\ f \neq 0$, $f \in F[x_1, \ldots x_n]$ such that $f(a_1, \ldots a_n) = 0$
$\forall\ a_i \in A$.

For elements $x_1, \ldots x_r$ in a ring R we write
$[x_1, \ldots x_r] = \pm x_{\pi(1)} \cdots x_{\pi(r)}$ where the sum runs over all the permutations π and the $+$ or $-$ sign is prefixed according to whether π is even or odd.

I. Kaplansky in [20] proves that in any algebra A of dimension $k - 1$ we have $[x_1, \ldots x_k] = 0$, $\forall\ x_i \in A$. Hence if A is an algebra of n by n matrices over a field and $r(n)$

denotes the smallest integer such that $[x_1, .. x_{r(n)}] = 0$, $r(n) \leqslant n^2 + 1$.

Let A be any Banach algebra, C_n the set of primitive ideals of P such that A/P is a k by k matrix algebra with $k \leqslant n$ and I_n the intersection of these ideals. Then A/I_n satisfies the identity $[x_1, .. x_{r(n)}] = 0$ and so does every primitive image of A/I_n. Hence C_n is a closed subset of $\mathrm{Prim}(A)$. It follows from proposition 2.1.9. that each of the primitive images of I_{r-1}/I_r is an r by r matrix algebra.

Let A be any C*-algebra satisfying a polynomial identity. The above defined chain of ideals reaches 0 in a finite number of steps. Hence we have constructed a finite composition series I_r with the property that every factor algebra has a Hausdorff structure space.

2.2.15. Theorem. A a C*-algebra such that every primitive image A/P is finite dimensional. Then A has a composition series $(I_\rho)_{\rho_0 \leqslant \rho \leqslant \alpha}$ such that each $I_{\rho+1}/I_\rho$ satisfies a polynomial identity.

Proof: Let C_n denote the subset of $\mathrm{Prim}(A)$ such that A/P has degree not greater than n. Then X is the union of a countable family of closed sets. Since X is of the second category (Theorem 2.2.13.) $\exists\ r$ say C_r such that $C_r^\circ \neq \varphi$. Let $U = C_r^\circ$. Let I denote the intersection of the primitive ideals comprising the complement of U. Since the latter is closed these are precisely the primitive ideals containing I. In particular we see that I is non-zero. By proposition 2.1.9. the primitive ideals in I itself are in one to one correspondence with the members of U. It follows that the primitive images of I are all of degree not greater than r. Hence I satisfies a polynomial identity; to be precise the identity for r by r matrices. This is the beginning of our composition series. The algebra A/I again satisfies the hypothesis of our theorem (any primitive image of A/I is a primitive image of A). We continue by transfinite induction.

2.2.16. Corollary. If A is a C*-algebra such that A/P is finite dimensional $\forall\ P \in \mathrm{Prim}(A)$ then $\mathrm{Prim}(A)$ has a non-void open Hausdorff subset.

Proof: In the remark before Theorem 2.2.15, it was observed that a C*-algebra B with a polynomial identity has a finite composition series such that all the factor algebras possess a T_2 structure space. In particular the first non-zero ideal of this series has a T_2 structure space and by Proposition 2.1.9., the latter is homeomorphic to an open subset of $\mathrm{Prim}(B)$. Hence if we combine this with Theorem 2.2.15 and another application of Proposition 2.1.9., we have the required result.

We are now able to prove our main theorem.

2.2.17. Theorem. A C.C.R. algebra has a composition series I_ρ, such that each $I_{\rho+1}/I_\rho$ has $\mathrm{Prim}(I_{\rho+1}/I_\rho)$ Hausdorff.

Proof: Let A be the algebra. Take a self-adjoint element $a \in A$ whose spectrum lies in $(0, 1]$.

For any $P \in \mathrm{Prim}(A)$, $Sp_{A/P}a(P)$ is a finite or countable set with at most 0 as a limit point because $a(P)$ can be realized as a self-adjoint positive definite completely continuous operator.

Let $p(t)$ be a continuous linear real valued function vanishing in a neighbourhood of 0 and satisfying $p(1) = 1$.

Putting $b = p(a)$ it can be seen that $\forall P \in \mathrm{Prim}(A)$ $Sp_{A/P}b(P)$ is finite lying between 0 and 1. Let B denote the intersection of the primitive ideals containing b or in other words $B = \{ P \in \mathrm{Prim}(A) ; b(P) = 0 \}, b(P) = 0 \Rightarrow p(a(P))=0$. Let $Y = \mathrm{Prim}(B)$. By Proposition 2.1.9., B is again C.C.R. and Y is in a natural way an open subset of $\mathrm{Prim}(A)$. It will now be shown that Y has an open Hausdorff subset.

Now $b(Q) \neq 0 \quad \forall Q \in Y$. For $n = 2,3, ..$ Let C_n be the set of $Q \in Y$ for which $Sp_{B/Q}b(Q) \subset \{ 0 \cup [\frac{1}{n}, 1] \}$ which is closed. By Lemma 2.1.10., C_n is closed. Also since the $Sp_{B/Q}b(Q)$ for each $Q \in X$ is finite, $Y = \cup C_n$. Hence by Theorem 2.2.13., one of the C's say $C_r^{\circ} = U \neq \phi$. Let $q(t)$ be a continuous real valued function satisfying $q(0) = 0$, $q(t) = 1$ for $t \geq \frac{1}{r}$ and put $c = q(b)$.

Then at every point of C_r , c maps into a non-zero self-adjoint idempotent and this is certainly true at all points of the closure V of U .

Let J be the intersection of the primitive ideals in B which comprise V . Let $D = B/J$. The space Prim(D) can be identified with V .

Let $e = c(J)$, then e is a self-adjoint idempotent not vanishing at any point of V .

Finally the algebra eDe has its structure space Prim(eDe) homeomorphic to V . (By Proposition 2.1.9.) Now it follows by Proposition 2.1.9. again, that any primitive ideal in eDe is of the form $R \cap eDe = eRe$ where R is primitive in D .

Thus the primitive homomorphic images of eDe, that is $eDe/eRe = eDe/R \cap eDe = eDe + R/R = e_1(D/R)e_1$, e_1 being the image of e mod R .

Now D/R is the algebra of all completely continuous operators on a Hilbert space H . Therefore $e_1(D/R)e_1$ is finite dimensional.

Hence all the primitive images of eDe are finite dimensional. Therefore the Corollary 2.2.16 is applicable and shows us that V has a non-void open T_2 subset say Z . Therefore $T = Z \cap U$ is a non-void open T_2 subset of Y . T is open in X as Y is open in X.

Let I be the intersection of the primitive ideals comprising the complement of T. Then I is a non-zero closed ideal in A whose structure space is homeomorphic to the T_2 space T. This is the beginning of our composition series and we continue with a similar treatment of A/I and continue by transfinite induction.

If A is a C.C.R. algebra whose locally compact structure space $Prim(A)$ is Hausdorff then Proposition 2.1.11. and Theorem 2.1.12., show that the representation of A on $Prim(A)$ gives us functions $\hat{A}$ with continuous norm vanishing at infinity. Hence by a theorem in M.A. Naimark [1], (as $\hat{A}$ is complete in the norm $\| \hat{a} \| = \sup_{P \varepsilon Prim(A)} \| a(P) \| = \| a \|$) $\hat{A}$ is closed under multiplication by $C^R(Prim(A))$.

Section 3. Representation of a G. C. R. and C. C. R. Algebra.

The representation theorem will be proved using the concept of continuous field of Banach spaces which will be defined now.

Let T be a topological space, $(E_t)_{t\varepsilon T}$ a family of Banach spaces over $\mathbb{C}$. A vector field x is any element belonging to $\prod_{t \varepsilon T} E_t$. That is $x(t) \varepsilon E_t$, $\forall t \varepsilon T$.

More generally if $Y \subset T$ then a vector field on Y is any element of $\prod_{t\in Y} E_t$.

2.3.1. Definition. Let T be a topological space. A continuous field $\mathfrak{B}$ of Banach spaces on T is a family $(E_t)_{t\in T}$ of Banach spaces, a subset $\Gamma \subset \prod_{t\in T} E_t$ of vector fields such that:

i) Γ is a complex vector subspace of $\prod_{t\in T} E_t$.

ii) $\forall\, t \in T$ the set $\{ x(t) ; x \in \Gamma \}$ is dense in E_t.

iii) $\forall\, x \in \Gamma$ the function $t \to \| x(t) \|$ is continuous.

iv) Let $x \in \prod_{t\in T} E_t$. If $\forall\, t \in T$, $\forall\, \epsilon > 0$, $\exists\, x' \in \Gamma$ such that $\| x(t) - x'(t) \| \leq \epsilon$ on a neighbourhood of t then $x \in \Gamma$.

The elements of Γ are called continuous vector fields with respect to Γ of $\mathfrak{B}$. Let $Y \subset T$ and $t_o \in Y$. A vector field x on Y is said to be continuous with respect to Γ at t_o if $\forall\, \epsilon > 0$, $\exists\, x' \in \Gamma$ such that $\| x(t) - x'(t) \| \leq \epsilon$ on a neighbourhood of t_o. It is said to be continuous on Y if it is continuous at every point of Y.

2.3.2. Proposition. Let $\mathfrak{B} = (E_t, \Gamma)$ be a continuous field of Banach spaces on T.

i) If $y \in \Gamma$ and $g : T \to \mathbb{C}$ continuous, then $g\,y \in \Gamma$.

ii) $\forall\, t_o \in T$ and $\forall\, \xi \in E_{t_o}$, $\exists\, x \in \Gamma$ such that $x(t_o) = \xi$.

See J. Dixmier [6].

2.3.3. Definition. $\mathfrak{B} = (E_t, \Gamma)$ a continuous field of Banach spaces on T, then $\Psi \subset \Gamma$ is said to be total if $\forall\, t \in T$ the set $\{ x(t) \,;\, x \in \Psi \}$ is total in E_t. That is the vector subspace which it generates is dense in E_t.

2.3.4. Proposition. Let $\mathfrak{B} = (E_t, \Gamma)$ be a continuous field of Banach spaces on T. Let $\Psi \subset \Gamma$ be a total subset and $\bar{\bar{\Psi}}$ the vector subspace of Γ generated by Ψ. For $x \in \prod_{t \in T} E_t$ the following properties are equivalent.

i) $x \in \Gamma$.

ii) $\forall\, t_0 \in T$ and $\forall\, \epsilon > 0$, $\exists\, x' \in \Gamma$ such that $\| x(t) - x'(t) \| \leq \epsilon$ on a neighbourhood of t_0.

ii') $\forall\, t_0 \in T$ and $\forall\, \epsilon > 0$, $\exists\, x' \in \bar{\bar{\Psi}}$ such that $\| x(t) - x'(t) \| \leq \epsilon$ on a neighbourhood of t_0.

iii) $\forall\, x' \in \Gamma$ the function $\| x(t) - x'(t) \|$ is continuous on T.

iii') $\forall\, x' \in \bar{\bar{\Psi}}$ the function $t \to \| x(t) - x'(t) \|$ is continuous.

Proof: (ii') $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (iii'). Evident.

(iii') $\Rightarrow$ (ii'). Suppose condition (iii') is satisfied. Let $t_0 \in T$ and $\epsilon > 0$. As $\bar{\bar{\Psi}}_{t_0}$ is dense in E_{t_0} $\exists\, x' \in \bar{\bar{\Psi}}$ such that $\| x(t_0) - x'(t_0) \| < \epsilon$. By (iii') one has $\| x(t) - x'(t) \| < \epsilon$ on a neighbourhood of t_0. Hence (ii').

The next proposition shows us under which conditions a subset of $\prod_{t\varepsilon T} E_t$ defines a continuous field of Banach spaces.

2.3.5. Proposition. Let T be a topological space and E_t, $t \varepsilon T$ a family of Banach spaces. If Ψ a subset of $\prod_{t\varepsilon T} E_t$ satisfies the axioms (i) - (iii) of Definition 2.3.1., then $\exists$ a unique subset Γ of $\prod_{t\varepsilon T} E_t$ containing Ψ characterized by (ii') such that $\mathfrak{B} = (E_t, \Gamma)$ is a continuous field of Banach spaces.

Proof: Follows from Proposition 2.3.4.

2.3.6. Definition. If $\mathfrak{B} = (E_t, \Gamma)$, $\mathfrak{B}' = (E_t', \Gamma')$ are two continuous fields of Banach spaces on T then an isometric isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}'$ is a family $\varphi = (\varphi_t)_{t\varepsilon T}$ such that

i) φ transforms Γ onto Γ'.

ii) Each φ_t is an isometric isomorphism of E_t onto E_t'.

We shall now consider the case when T is a locally compact Hausdorff space, $\mathfrak{B} = (E_t, \Gamma)$ a continuous field of Banach spaces on T, T' the compact space obtained from T by adjoining ω a point at infinity. Put $E_t' = E_t$, $\forall\ t \varepsilon T$, $E_\omega' = 0$. Let Γ' be the set of $x \varepsilon \prod_{t\varepsilon T'} E_t'$ such that $x|T \varepsilon \Gamma$ and $\| x(t) \| \to 0$ at infinity on T.

2.3.7. Proposition. $\mathfrak{B}' = (E_t', \Gamma')$ is a continuous field of Banach spaces on T' and $\mathfrak{B}'|T = \mathfrak{B}$.

See J. Dixmier [6].

2.3.8. <u>Definition</u>. Let T be a topological space. A continuous field of C*-algebras on T is a continuous field (A_t, θ) of Banach spaces on T with each A_t being a C*-algebra and θ closed under multiplication and self-adjoint. A continuous field of elementary C*-algebras is a continuous field (A_t, θ) of C*-algebras such that each A_t is elementary.

2.3.9. <u>Proposition</u>. Let $\mathfrak{B} = (A_t, \theta)$ be a continuous field of Banach spaces on T. $\mathfrak{B}$ is a continuous field of C*-algebras iff $\exists$ a total subset Ψ of θ which is closed under multiplication and involution.

<u>Proof</u>: $\tilde{\Psi}$ the vector subspace of θ generated by Ψ is closed under multiplication and involution. The result follows from Proposition 2.3.4.

We shall now show how $\mathfrak{I} = (A_t, \theta)$ a continuous field of C*-algebras on T a locally compact Hausdorff space, defines a C*-algebra. Let $A = \{ x \in \theta \, ; \, \| x(t) \|$ vanishes at infinity$\}$. Then A is an involution subalgebra of θ. $\forall\, x \in A$ define $\| x \| = \sup_{t \in T} \| x(t) \|$ then A becomes a C*-algebra, and the mapping $x \to x(t)$ of A into A_t is a *-homomorphism.

2.3.10. <u>Lemma</u>. Let I be a closed ideal in a C*-algebra A. $\forall\, t \in T$ let $I_t = \{ x(t) \, ; \, x \in I \}$ then $I = \{ x \in A \, ; \, x(t) \in I_t, \ \forall\, t \in T \}$.

See M.A. Naimark [27].

2.3.11. Theorem. Let T be a locally compact Hausdorff space, $\mathfrak{J} = (A_t, \theta)$ a continuous field of C*-algebras on T, A the C*-algebra defined by $\mathfrak{J}$. $\forall\ t \in T$ and $\forall\ S \in \hat{A}_t$, let Q_s be the irreducible representation $x \to S(x(t))$ of A. Then $S \to Q_s$ is a bijection of the set sum of A_t onto $\hat{A}$.

Proof. Let $S \in \hat{A}_t$, $S' \in \hat{A}_{t'}$. If $t \neq t'$ $\exists$ a continuous complex function f on T such that $f(t) = 1$, $f(t') = 0$. Let $x \in A$ such that $S(x(t)) \neq 0$, then

$Q_s(fx) = S(x(t)) \neq 0$, $Q_{s'}(fx) = S'(0) = 0$. Hence Q_s and $Q_{s'}$ are not equivalent. Therefore the mapping $S \to Q$ is one to one. Let $Q \in \hat{A}$, and I the kernel of Q. Let $I_t = \{ x(t) ; x \in I \}$. Let $Y = \{ t \in T ; I_t \neq A_t \}$. By Lemma 2.3.10, Y is non-void. Suppose Y contains two distinct points t_1 and t_2. Let U_1, U_2 be the disjoint neighbourhoods of t_1 and t_2. Let K_i be the closed ideal in A consisting of the $x \in A$ which vanish on the complement of U_i. Then $I \supset 0 = K_1 K_2$ therefore by Theorem 1.1.13, either K_1 or K_2 is contained in I. Say $K_1 \subset I$, then $I_{t_1} \supset K_{1t_1} = A_{t_1}$ which is absurd as $t_1 \in Y$. Hence $Y = \{ t_0 \}$. By Lemma 2.3.10., $I = \{ x \in A ; x(t_0) \in I_{t_0} \}$. Therefore Q is composed of the *-homomorphism $x \to x(t_0)$ and an irreducible representation of A.

2.3.12. Corollary. Let T be a locally compact Hausdorff space, $\mathfrak{J} = (A_t, \theta)$ a continuous field of non-zero elementary C*-algebras,

A the C*-algebra defined by $\mathfrak{J}$. $\forall\, t \in T$, let Q_t be the irreducible representation of A composed of the *-homomorphism $t \to x(t)$ and the irreducible representation unique up to equivalence of A_t. Then the map $t \to Q_t$ is a homeomorphism of T onto $\hat{A}$.

Proof: This mapping is one to one by Proposition 2.3.11. Now let $Y \subset T$, $t_\circ \in T$. The intersection of the kernels of the Q_t, $t \in Y$ denoted by $I(Y) = \{ x \in A ; x(t) = 0 \text{ on } Y \}$. Now $Q_{t_\circ}$ belongs to the closure of the set $\{ Q_t ; t \in Y \}$ iff $I(\{ t_\circ \}) \supset I(Y)$. If $t_\circ \in \bar{Y}$ then $I(\{ t_\circ \}) \supset I(Y)$. If $t_\circ \notin \bar{Y}$ then $\exists$ a continuous complex valued function on T equal to 1 on $t_\circ$ and 0 on Y. Hence an $x \in A$ non-zero on $t_\circ$ and zero on $Y \Rightarrow I(\{ t_\circ \}) \not\supset I(Y)$.

Hence one can identify T and $\hat{A}$.

We shall now state a non-commutative type of Stone-Weierstrass theorem which will enable us to prove the representation theorem.

2.3.13. Definition. Let E, F be dual vector spaces with $A \subset E$, $B \subset F$. A is said to separate B if for every couple x, y of distinct points of B, $\exists\, a \in A$ such that $\langle a , x \rangle \neq \langle a , y \rangle$.

2.3.14. Glimm's Stone-Weierstrass Theorem. Let A be a C*-algebra with an identity and B a sub-C*-algebra containing the identity. If

B separates the weak closure of $P(A)$ (that is $\overline{P(A)}$) then $B = A$.

See J. Glimm [11].

The following corollary generalizes Glimm's theorem to the case that there is no identity.

2.3.15. Corollary. A a C*-algebra, B a sub C*-algebra separating $\overline{P(A)} \cup \{ 0 \}$ then $B = A$.

Proof: Let $\tilde{A}$ be the C*-algebra obtained from A by adjoining an identity. That is $\tilde{A} = A + \mathbb{C} e$. Let $\tilde{B} = B + \mathbb{C} e$ and let $f \in \overline{P(\tilde{A})}$. Then f is the weak limit of pure states f_α on $\tilde{A}$. If $f \mid A$ is non-zero, one may suppose that the $f_\alpha \mid A$ are non-zero. These are pure states of A therefore $f \mid A \in \overline{P(A)}$. Hence $f \mid A \in \overline{P(A)} \cup \{0\}$ in every case. Then if f, g are two distinct elements of $\overline{P(\tilde{A})}$, $f \mid A$ and $g \mid A$ are distinct and belong to $\overline{P(A)} \cup \{ 0 \}$, hence are separated by B. Thus $\tilde{B}$ separates $\overline{P(\tilde{A})}$ and by Theorem 2.3.14, we have $\tilde{B} = \tilde{A} \Rightarrow B = A$.

2.3.16. Corollary. Let T be a locally compact Hausdorff space, $\mathfrak{J} = (A_t, \theta)$ a continuous field of C*-algebras on T, A the C*-algebra defined by $\mathfrak{J}$, B a sub C*-algebra of A such that for every $t_1 \in T$, $t_2 \in T$, $\xi_1 \in A_{t_1}$, $\xi_2 \in A_{t_2}$ ($\xi_1 = \xi_2$ if $t_1 = t_2$) $\exists\, x \in B$ with $x(t_1) = \xi_1$, $x(t_2) = \xi_2$ then $B = A$.

<u>Proof</u>: By Proposition 2.3.7., we may adjoin the infinite point to T and suppose that T is compact.

Let $f \in \overline{P(A)}$. Then f is the weak limit of pure states f_α of A. By Theorem 2.3.11., $\exists\ t_\alpha \in T$ and a pure state g_α of A_{t_α} such that $f_\alpha(x) = g_\alpha(x(t_\alpha))$, $\forall\ x \in A$. One may suppose that the t_α tend to a point $t \in T$. Let $x \in A$ such that $x(t) = 0$. Now $\| x(t_\alpha) \| \to \| x(t) \| = 0 \Rightarrow g_\alpha(x(t_\alpha)) \to 0$ hence $f(x) = 0$. Hence $\exists$ a positive functional g on A_t such that $f(x) = g(x(t))$ $\forall\ x \in A$.

Let f_1, f_2 be two distinct points of $\overline{P(A)} \cup \{0\}$. We shall show that B separates f_1 and f_2 and this corollary will then result from Corollary 2.3.15.

Let t_1, $t_2 \in T$ and g_i a positive functional on A_{t_i} such that $f_i(x) = g_i(x(t_i))$ $\forall\ x \in A$. At least one of the two functionals g_1 and g_2 is non-zero, for example g_1. If $t_1 = t_2$, $g_1 \neq g_2$ therefore $\exists\ \xi \in A_{t_1}$ such that $g_1(\xi) \neq g_2(\xi)$. Since $\exists\ x \in B$ such that $x(t_1) = \xi$, then $f_1(x) \neq f_2(x)$. If $t_1 \neq t_2$, $\exists\ y \in B$ such that $y(t_2) = 0$, $g_1(y(t_1)) \neq 0$ hence $f_1(y) \neq f_2(y)$.

2.3.17.<u>Definition</u>. A C*-algebra is said to have a <u>continuous trace</u> if it is C.C.R., its structure space $\hat{A}$ is Hausdorff and $\forall\ T \in \hat{A}$, $\exists\ a \in A$ and a neighbourhood of T, U_T such that $\forall\ S \in U$, S_a is a one dimensional projection in H_S.

For Q a linear operator, dim Q denotes the dimension of the closure of the range of Q .

The phrase, "Continuous trace", is used because it can be shown that if A is a C*-algebra with continuous trace, then the map $S \to \mathrm{Trace}(S_a)$ is continuous on $\hat{A}$, $\forall\, a \in A$ such that $\exists$ an integer n with $\dim(T_a) \leq n$, $\forall\, T \in \hat{A}$.

See J.M.G. Fell [9].

2.3.18. Theorem. Every G.C.R. algebra has a composition series $(I_\rho)_{0 \leq \rho \leq \alpha}$ such that $I_{\rho+1}/I_\rho$ is a C*-algebra with continuous trace.

Proof: By Proposition 2.1.5., $\hat{A}$ and Prim(A) are homeomorphic hence by Theorem 2.2.17, it is sufficient to assume that A is C.C.R. with $\hat{A}$ Hausdorff and to show that A has a non-zero closed ideal I with continuous trace.

By a result in I. Kaplansky [22], there is a non-zero positive element a of A such that aAa is commutative. Let I be the smallest closed ideal containing a . Since A is C.C.R. and Hausdorff, the same is true for I . The commutativity of $aAa \Rightarrow \dim(T_a) \leq 1$, $\forall\, T \in \hat{A}$. If $T \in I$, $a \notin \mathrm{Kernel}(T)$. Hence $\forall\, T \in I$, T_a is a positive multiple of a one dimensional projection. Thus if $S \in \hat{I}$, we may apply to the element a some suitable real continuous function f so that $T_{f(a)}$ is a one dimensional projection throughout a neighbourhood of S .

Hence I has continuous trace.

We now have the representation theorem for G.C.R. algebras

2.3.19. Theorem. A G.C.R. algebra C has a composition series $(I_\rho)_{0\leqslant\rho\leqslant\alpha}$ such that each factor $I_{\rho+1}/I_\rho$ is isometrically-*-isomorphic to the C*-algebra of all the vector valued functions defined on a locally compact Hausdorff space T, (Its structure space), which are continuous in the norm and vanish at infinity.

Proof: C is a G.C.R. algebra, hence by Theorem 2.3.18., $\exists$ a composition series $(I_\rho)_{0\leqslant\rho\leqslant\alpha}$ such that $I_{\rho+1}/I_\rho$ is a C*-algebra with continuous trace. That is $B = I_{\rho+1}/I_\rho$ is a C.C.R. algebra and $T = \mathrm{Prim}(B)$ is a locally compact Hausdorff space. Every $b \in B$ can be represented as a vector valued function, or vector field $b' \in \prod_{t\in T} B_t$ where B_t is B mod t and $b'(t)$ is the homomorphic image of b mod t. By Theorem 2.1.12. and Proposition 2.1.11., b' is a vector valued function continuous in the norm and vanishing at infinity. This functional representation preserves the norm. That is

$$\| b' \| = \sup_{t\in T} \| b'(t) \| = \| b \| \quad \text{(By Proposition 2.1.7.)}$$

As B is semi-simple, this representation is an isomorphism. Hence B is isometrically-*-isomorphic to $B' = \{ b';\ b \in B \}$, a C*-algebra of vector valued functions with continuous norm vanishing at infinity.

Set $A = \{ x \in \prod_{t \in T} B_t ; \ \| x(t) \|$ is continuous and vanishes at infinity $\}$. Then A is an involution subalgebra of $\prod_{t \in T} B_t$ and with norm $\| x \| = \sup_{t \in T} \| x(t) \| < \infty$, A becomes a C*-algebra.

A satisfies conditions (i), (ii) and (iii) of Definition 2.3.1., hence by Proposition 2.3.5., $\exists$ a unique subset Γ of $\prod_{t \in T} B_t$ containing A such that $\mathfrak{Z} = (B_t , \Gamma)$ is a continuous field of Banach spaces. By Proposition 2.3.9., $\mathfrak{Z} = (B_t, \Gamma)$ is a continuous field of C*-algebras and therefore defines a C*-algebra A'. Now $A' \supset A$ by definition of A' and $A' \subset A$ by definition of A. Hence $A' = A$. Now B' is a sub C*-algebra of A and B' satisfies the following condition:

(i) For any $t_1 \in T$, $t_2 \in T$, $\xi_1 \in B_{t_1}$, $\xi_2 \in B_{t_2}$ ($\xi_1 = \xi_2$ if $t_1 = t_2$) $\exists \ x \in B$ such that $x(t_1) = \xi_1$, $x(t_2) = \xi_2$ since the ideal $t_1 + t_2$ is dense in B (Otherwise there would exist a primitive ideal containing $t_1 + t_2$ which would contradict the maximality of t_1 and t_2) hence by Proposition 2.1.4., $t_1 + t_2 = B$. Let $a'_1(t_1) = \xi_1$ and $a_2(t_2) = \xi_2$ where $a_1, a_2 \in B$. Then $\exists \ b_1 \in t_1, b_2 \in t_2$ such that $a_1 - a_2 = b_1 - b_2$. Let $x = a_1 - b_1 = a_2 - b_2$, then $x(t_1) = \xi_1$ and $x'(t_2) = \xi_2$. Now $B' \subset A = A'$ hence by Corollary 2.3.16., $B' = A$. ______

2.3.20. <u>Corollary</u>. Every C.C.R. algebra has a composition series satisfying the conditions of the previous theorem.

<u>Notes and Remarks</u>.

Theorems 2.1.12 and 2.1.13 are due to I. Kaplansky [23] and in fact J.M.G. Fell [8] generalized Theorem 2.1.13 to the following case. "A a C*-algebra all of whose representations have the same finite dimension then A is Hausdorff."

The results about G.C.R. and C.C.R. algebras are due to J. Dixmier [6] and the main theorems of Section 2, namely Theorems 2.2.13 and 2.2.17 are due to I. Kaplansky [23].

For the definitions and results of a continuous field of Banach spaces, see J. Dixmier [6] and J. Dixmier and A. Douady [7]. Corollary 2.3.15 the generalization of Glimm's Stone-Weierstrass theorem was first proved by J. Tomiyama [31]. Corollary 2.3.16 is due to J. Dixmier [6] and a similar result is proved by J. Tomiyama [31]. The result about C*-algebras with continuous trace, namely Theorem 2.3.18 is due to J.M.G. Fell [9]. Finally, Theorem 2.3.19 was constructed using the results of J. Dixmier [6], J.M.G. Fell [9] and J.Tomiyama [31].

CHAPTER III

Representations of Rings with Hausdorff Structure Spaces by Continuous Functions.

Let A be a ring, Prim(A) its structure space and $B = \bigcup_{x \varepsilon \mathrm{Prim}(A)} A_x$ where $A_x = A \bmod x$. In Chapter II, when A_x was a Banach algebra over the complexes $\forall\ x \varepsilon \mathrm{Prim}(A)$ a notion of "continuity" for a vector valued function $\theta \varepsilon \prod_{x \varepsilon \mathrm{Prim}(A)} A_x$ was defined as B had no topological structure. In this chapter a topology will be introduced on B such that the induced topology on A_x coincides with the original norm topology on A_x in the case that the A_x are Banach algebras. The concept of a vector valued function or vector field will be replaced by a section.

In Section I we shall prove a representation theorem for a biregular ring which has a purely algebraic structure. For any biregular ring A , Prim(A) is Hausdorff and Theorem 3.1.11. shows us that every biregular ring is isomorphic to the ring of global sections with compact supports in a sheaf of simple rings with identity defined on Prim(A) . In Theorem 3.1.12. the converse will be proved.

In Section 2, a similar type of representation theorem will

will be proved for an n - dimensional homogeneous C*-algebra, (over the complexes), in the setting of a fibre bundle and using the Stone-Weierstrass Theorem obtained in Chapter II. Theorem 3.2.12. shows us that every n - dimensional homogeneous C*-algebra is isometrically-*-isomorphic to the C*-algebra of all continuous sections which vanish at infinity in a fibre bundle with base space $\hat{A}$ which is homeomorphic to Prim(A) . In Theorem 3.2.11. the converse will be proved. Proposition 3.2.15. will then show us the relationship between the spectrum of an element of the algebra and its image under the above representation.

In Section 3, the question is considered as to what type of representation theorem can be proved for a C*-algebra whose structure space is not necessarily Hausdorff. Initially it will be shown how to define a topology on $\bigcup_{t\in T} A_t$ where $\mathfrak{J} = (A_t, \theta)$ is a continuous field of Banach spaces and T is a topological space. Then the results of J. Tomiyama [31], who considered this question, will be stated, but no proofs will be given as this question in a general setting will be discussed in Chapter IV.

Section I. Representation of Biregular Rings using the Theory of Sheaves.

3.1.1. Definition. A ring is called biregular if every principal ideal is generated by a central idempotent.

Example. $C_0(X, R)$ the ring of all continuous functions with compact supports defined on a locally compact totally disconnected Hausdorff space X into a semi-simple discrete ring R is a biregular ring.

D.R. Morrison [26] has proved the following equivalent statements for a biregular ring R (where R is with local unit if R is the union of its ideals with unit).

i. R is biregular.

ii. Every principal ideal of R is with unit.

iii. Every ideal of R is a ring with local unit.

N. Jacobson [15] proved that for a biregular ring A the concepts of primitive, maximal and maximal modular coincide. $M(A)$ will denote the set of maximal ideals with the hull-kernel topology. Then $M(A)$ becomes a locally compact, totally disconnected Hausdorff space and hence is zero-dimensional. As every biregular ring is semi-simple and is primitive iff it is a simple ring with an identity, A/x is simple and has an identity $\forall\, x \in M(A)$.

3.1.2. Definition

A sheaf of rings, (or simple rings) with identity is a triple $\langle S, \pi, X \rangle$ where:

i. S and X are topological spaces, π a continuous map from S onto X.

ii. Each point in S has an open neighbourhood which is mapped homeomorphically onto an open set in X under π . $\pi^{-1}(x)$ is called the <u>stalk</u> over x and is a ring (or simple ring) with identity.

It follows from (ii) that the topology induced on the stalks is the discrete topology.

iii. The function $(s,t) \to s + t$ and $(s,t) \to st$ from the set $\{(s,t) \; ; \; \pi(s) = \pi(t)\}$ into S is continuous.

iv. The function which assigns to every $x \in X$ the identity $1(x)$ of $\pi^{-1}(x)$ is continuous.

The zero of $\pi^{-1}(x)$ will be denoted by $0(x)$ and it follows from (i), (ii) and (iii) of Definition 3.1.2. that the map $x \to 0(x)$ is continuous.

3.1.3. <u>Definition</u>. A <u>section</u> σ is a continuous function from an open set $U \subset X \to S$ such that $\pi_\circ\sigma$ is the identity. $\Gamma(U,S)$ will denote the set of all sections over U and it is a ring with identity under the point-wise operations. If $U = X$ then we write $\Gamma(S)$ for $\Gamma(X,S)$ and elements belonging to $\Gamma(S)$ are called <u>global sections.</u> The support of an element $\sigma \in \Gamma(U,S)$ is the set $\{ x \in X \; ; \; \sigma(x) \neq 0(x) \; , \; x \in U \}$. The supports of sections are closed and if $\sigma_1(x_\circ) = \sigma_2(x_\circ)$ then for some neighbourhood of $x_\circ$ say $U_{x_\circ}$, $\sigma_1|_{x_\circ} = \sigma_2|U_{x_\circ}$. $\Gamma_\circ(S)$ denotes the subring of $\Gamma(S)$ consisting of global sections with compact supports.

For A a biregular ring let $S(A) = \bigcup_{x \in M(A)} A/x$ and define $\pi : S(A) \to M(A)$ by $\pi(s) = x$ where $s \in A/x$. For $a \in A$, $\hat{a}$ denotes the function $: M(A) \to S(A)$ defined by $\hat{a}(x) = a \bmod x$. Let $\hat{A} = \{ \hat{a} ; a \in A \}$. The map $A \to \hat{A}$ is a ring isomorphism as A is semi-simple and $\{ x \in M(A) ; \hat{a}(x) \neq x \}$ is compact open $\forall a \in A$.

R. Arens and I. Kaplansky in [1] proved that a biregular ring A contains the characteristic function of any open compact set $E \subset M(A)$ (where e is the characteristic function of E if $\hat{e}(P) = 1$, $\forall P \in E$ and $\hat{e}(P) = 0$, $\forall P \notin E$). Hence we have the following Lemma.

3.1.4. <u>Lemma</u>. If $U \subseteq M(A)$ is compact open then $\exists$ a central idempotent $e \in A$ such that the support of $\hat{e}$ is exactly U.

3.1.5. <u>Lemma</u>. Let $< S , \pi , X >$ be a sheaf of simple rings with identity over a locally compact, totally disconnected Hausdorff space X. Then:

i. For I an ideal of $\Gamma_0(S)$, $x \in X$ and $\sigma \in I$ with $\sigma(x) \neq 0(x)$ then I contains the characteristic function of some compact open neighbourhood U of x.

ii. For $\sigma \in \Gamma(S)$, $W = \{ x ; x \in X , \sigma(x) \neq 0(x) \}$ is open.

<u>Proof</u>: (i) $\pi^{-1}(x)$ is simple hence there are $2n$ elements $s_1 , \dots , s_n ; t_1 , \dots , t_n \in \pi^{-1}(x)$ such that

$s_1\sigma(x)t_1 + \dots + s_n\sigma(x)t_n = 1(x)$. As π is a local homeomorphism and the topology of X has a basis of open compact sets $\exists$ global sections $\alpha_i, \beta_i \varepsilon \Gamma_0(S)$ such that $\alpha_i(x) = s_i$ and $\beta_i(x) = t_i$, $i = 1, \dots n$. The set
$U = \{ y \varepsilon X ; \alpha_1(y)\sigma(y)\beta_1(y) + \dots + \alpha_n(y)\sigma(y)\beta_n(y) = 1(y)\}$
is compact open as the set of points at which two sections agree is open. After multiplying α_i and β_i by the characteristic function of U and renaming them, we may assume that each α_i and β_i has the compact open set U as its carrier. Hence I contains $\alpha_1\sigma\beta_1 + \dots + \alpha_n\sigma\beta_n$.

(ii) If $\sigma(x) \neq 0(x)$ for some $x \varepsilon X$, let the ideal I of part (i) be the principal ideal generated by $\sigma e \varepsilon \Gamma_0(S)$ where e is the characteristic function of some compact open neighbourhood of x . Then by part (i) there are $\alpha_i, \beta_i \varepsilon \Gamma_0(S)$ and a compact open neighbourhood U of x such that
$\alpha_1\sigma e\beta_1 + \dots + \alpha_n\sigma e\beta_n$ is the characteristic function of U .
In particular, $\sigma(y) \neq 0(y)$, $\forall\, y \varepsilon U$ hence W is open.

We shall now show that for A a biregular ring $S(A)$ can be given a topology so that $<S(A) , \pi , M(A)>$ becomes a sheaf of simple rings with identity.

3.1.6. <u>Lemma</u>. Suppose that $a, b \varepsilon A$ and that U, V are compact open sets in $M(A)$. If $s \varepsilon \hat{a}(U) \cap \hat{b}(V)$ then $\exists$ a compact open neighbourhood W of $\pi(s)$ with $W \subseteq U \cap V$ such that $\hat{a}$ and $\hat{b}$ are equal on W. In particular $\hat{a}(W) \subseteq \hat{a}(U) \cap \hat{b}(V)$.

Proof: If $\pi(s) = x$ then $\hat{a}(x) = \hat{b}(x) = s$.
$\{ x \in M(A) ; (\hat{a} - \hat{b})(x) = 0(x) \}$ is open. Since the compact open neighbourhoods of x form a basis for all neighbourhoods $\exists$ a compact open neighbourhood W of x such that $\hat{a}|W = \hat{b}|W$ and $W \subseteq U \cap V$.

3.1.7. Lemma. The set $\mathfrak{B} = \{ \hat{a}(u) ; a \in A , U$ compact open in $M(A) \}$ is a basis for a topology on $S(A)$. $\forall\, a \in A$ the function $\hat{a} : M(A) \to S(A)$ is continuous and open.

Proof: If $s \in S(A)$ then $\exists\, a \in A$ such that $a + \pi(s) = s$ that is $\hat{a}(\pi(s)) = s$. Hence $S(A) = \cup \mathfrak{B}$. That $\mathfrak{B}$ is a basis for a topology follows from Lemma 3.1.6.

3.1.8. Lemma. Let $g(x_1, .., x_n)$ be any non-commutative polynomial with integral coefficients in n indeterminates. The map of the subset $\{ < s_1, .., s_n > ; s_1, .., s_n \in S(A)$ and $\pi(s_1) = \ldots = \pi(s_n) \}$ of the n fold product of $S(A)$ into $S(A)$ given by $< s_1, .. s_n > \to g(s_1, .. s_n)$ is continuous and open. In particular the ring operations in $S(A)$ are continuous.

Proof: Let $\hat{a}(U)$ be a typical open neighbourhood of $g(s_1, .., s_n)$ with $a \in A$, U compact open in $M(A)$, and $\pi(s_1) = .. = \pi(s_n) = x \in U$. Choosing $a_1, .., a_n \in A$ with

$\hat{a}_j(x) = s_j$ for all j, by Lemma 3.1.6. there is a compact open set W with $x \in W \subseteq U$ such that $g(\hat{a}_1, \ldots, \hat{a}_n)|W = \hat{a}|W$. Hence $\{ < \hat{a}_1(y), \ldots, \hat{a}_n(y) >; y \in W \}$ is an open neighbourhood of $< s_1, \ldots, s_n >$ which is mapped onto $\hat{a}(W)$, an open subset of $\hat{a}(U)$.

3.1.9. <u>Proposition</u>. A a biregular ring, then $< S(A), \pi, M(A) >$ is a sheaf of simple rings with identity.

<u>Proof</u>: To verify condition (ii) of Definition 3.1.2. that π is a local homeomorphism, let $s \in S(A)$ and let U be any compact open neighbourhood of $\pi(s)$ in $M(A)$. Choose an $a \in A$ such that $\hat{a}(\pi(s)) = s$. If $\mathfrak{B}(U) = \{ W ; W$ compact open in $U \}$ denotes the basis for the induced topology on U then $\mathfrak{B}(U,a) = \{ \hat{a}(W) ; W \in \mathfrak{B}(U) \}$ is the basis for the topology induced on $\hat{a}(U)$ by the topology of $S(A)$. The map $\pi|\hat{a}(U) : \hat{a}(U) \to U$ is injective and surjective and induces an injective and surjective map $\mathfrak{B}(U, a) \to \mathfrak{B}(U)$. Hence $\pi|\hat{a}(U)$ is a homeomorphism of $\hat{a}(U)$ onto U.

Condition (ii) is satisfied by Lemma 3.1.8.

We shall now show that condition (iv) that the map $M(A) \to S(A)$, $x \to 1(x)$ is continuous, is satisfied. Take any compact open subset U of $M(A)$. By Lemma 3.1.4., $\exists$ a central idempotent $e \in A$ such that U is the exact support of

$\hat{e}$. But $\hat{e}(u)$, $u \varepsilon U$, being a non-zero idempotent in the simple ring $\pi^{-1}(u)$, is necessarily the identity of that ring. Hence $\hat{e}|U$ is the restriction of the map $x \to 1(x)$ to U . The latter is continuous and open since the former is.

We shall now prove the representation theorem for a biregular ring. We first need the following lemma.

3.1.10.Lemma. If $\sigma \varepsilon \Gamma(S(A))$ is a global section and U an open compact subset of $M(A)$ then $\exists$ $a \varepsilon A$ such that $\hat{a}|U = \rho|U$.

Proof: By Lemma 3.1.7. $\hat{A} \subset \Gamma_o(S(A))$. For each point $x \varepsilon M(A)$, $\exists$ a compact open neighbourhood $W(x)$ and an element $a_x \varepsilon A$ such that $\hat{a}_x|W(x) = \sigma|W(x)$. U can be represented as a disjoint union $U = U_1 \cup \ldots \cup U_n$ of a finite number of compact open subsets $U_1, \ldots, U_n$ every one of which is completely contained in some set $W(x)$. We may therefore assume that we have n elements $a_1, \ldots, a_n \varepsilon A$ such that $\hat{a}_i|U_i = \sigma|U_i$. By Lemma 3.1.4. we have central idempotents $e_1, \ldots, e_n$ whose supports are exactly the sets U_i respectively. After replacing the a_i with $a_i e_i$ and renaming them, we may assume that U_i is the exact carrier of a_i . Now if $a = a_i + \ldots + a_n$, then $\hat{a}|U = \sigma|U$.

3.1.11.Theorem. A a biregular ring then A is isomorphic to the ring $\Gamma_o(S(A))$ of global sections of $< S(A) , \pi , M(A) >$ with compact support.

<u>Proof</u>: $\hat{A} \subseteq \Gamma_o(S(A))$ and by Lemma 3.1.10. $\Gamma_o(S(A)) \subseteq \hat{A}$. Hence $\Gamma_o(S(A)) = \hat{A}$.

We shall now prove the converse.

3.1.12. <u>Theorem</u>. Let $< S , \pi , X >$ be a sheaf of simple rings with identity over a locally compact totally disconnected Hausdorff space. Then $\Gamma_o(S)$ is a biregular ring whose maximal ideal space is naturally homeomorphic to X . The ring $\Gamma_o(S)$ has an identity iff X is compact.

<u>Proof</u>: Let $\sigma \varepsilon \Gamma_o(S)$ and U the support of σ . By (ii) of Lemma 3.1.5., U is compact open. Hence if $k : X \to S$ is the characteristic function of U (that is $k(x) = 1(x)$ if $x \varepsilon U$ and $k(x) = 0(x)$ if $x \notin U$) then k is a central idempotent in the ring $\Gamma_o(S)$. The principal ideal (k) generated by k contains (σ) . Next we shall show that $k \varepsilon (\sigma)$. By (i) of Lemma 3.1.5. we have that U is a disjoint union $U = C_1 \cup .. \cup C_n$ of compact open subsets C_i of X such that the ideal (σ) contains the characteristic function of each C_i and hence also k . Thus $(\sigma) = (K)$ and $\Gamma_o(S)$ is biregular.

For any ideal I of $\Gamma_o(S)$, let $N(I) = \{ x \varepsilon X ; \sigma(x) = 0(x) , \forall \sigma \varepsilon I \}$. By Lemma 3.1.5.

part (i), if $x \notin N(I)$ then I contains the characteristic function of some compact open neighbourhood of x. Hence if $N(I) = \Phi$ then I contains the characteristic function of every compact open subset of X and $I = \Gamma_o(S)$. Thus $N(I) \neq \Phi$ if I is proper. Since the characteristic functions separate points of X, $N(I)$ cannot contain more than one point if I is maximal.

Conversely, suppose that $N(I) = \{x\}$, then I is necessarily the ideal consisting of all the $\sigma \in \Gamma_o(S)$ which vanish at x. For if $\sigma(x) = 0(x)$ and W is the support of σ then $x \notin W$. But by Lemma 3.1.5. part (i) I contains the characteristic function of any compact open set not containing x and hence $\sigma \in I$. Therefore the kernel of the map $\sigma \to \sigma(x)$ is exactly I and as $\pi^{-1}(x)$ is simple, I is maximal.

Denoting the maximal ideal space of $\Gamma_o(S)$ by $M(\Gamma_o(S))$ the mapping $\varphi : M(\Gamma_o(S)) \to X$ which assigns to a maximal ideal I the element x with $N(I) = \{x\}$ is injective and surjective. In order to show that φ is a homeomorphism, it suffices to show that φ induces a one to one correspondence between the compact open sets of $M(\Gamma_o(S))$ and those of X. According to Lemma 3.1.4. for any compact open subset C of $M(\Gamma_o(S))$ there exists a central idempotent $k \in \Gamma_o(S)$ such that C consists precisely of all the maximal ideals not containing k. But every central idempotent of $\Gamma_o(S)$ is the characteristic function of some compact open subset U of X. Hence $\varphi(C) = U$ and since the characteristic function of some compact open subset U of X is in $\Gamma_o(S)$ it thus determines a compact open set $C \subseteq M(\Gamma_o(S))$ with $\varphi(C) = U$. Thus $M(\Gamma_o(S))$ is homeomorphic to X. Clearly $\Gamma_o(S)$ has an identity iff X is compact.

Section 2. The Representation of Homogeneous C*-Algebras.

Let T be a locally compact Hausdorff space called the base space and $\forall\, t \in T$ let A_t be a complex C*-algebra. A vector field is a function $x \in \prod_{t\in T} A_t$.

3.2.1. Definition. A full algebra of vector fields is a family A of vector fields defined on T such that:

i. A is an involution algebra.

ii. $\forall\, x \in A$ the function $t \to \| x(t) \|$ is continuous and vanishes at infinity.

iii. $\forall\, t \in T$, $\{ x(t) ; x \in A \}$ is dense in A_t.

iv. A is complete in the norm $\| x \| = \sup_{t\in T} \| x(t) \|$.

Hence A is a C*-algebra. As in Chapter II, a vector field x on T is said to be continuous with respect to A if $\forall\, t \in T$, $\forall\, \epsilon > 0$, $\exists\, x' \in A$ such that,

$\| x'(t) - x(t) \| \leq \epsilon$, $\forall\, t$ belonging to some neighbourhood of t .

3.2.2. Definition. A full algebra of vector fields A is said to be maximal if it is equal to the set of all continuous vector fields with respect to itself which vanish at infinity.

This first theorem is needed to show under which conditions the structure space of a full algebra is homeomorphic to the base space.

3.2.3. Theorem. A a maximal full algebra of vector fields with base space T such that each A_t is elementary, then $\hat{A}$ is homeorphic to T .

Proof. $\mathfrak{J} = (A_t, A)$ is a continuous field of C*-algebras and the C*-algebra defined by $\mathfrak{J}$ is equal to A . Hence by Corollary 2.3.12. $\hat{A}$ is homeorphic to T .

A a full algebra of vector fields on T is said to be separating if for any $s, t \in T$, $\alpha \in A_s$ and $\beta \in A_t$, $\exists\ x \in A$ such that $x(s) = \alpha$, $x(t) = \beta$.

We shall now prove the type of Stone-Weierstrass theorem which will be needed to prove the representation theorem , (Theorem 3.2.12.).

3.2.4. Stone-Weierstrass Theorem.

A a full separating algebra of vector fields on T a locally compact Hausdorff space , the base space, with component algebras A_t is maximal.

Proof: By Propositions 2.3.5 and 2.3.9, A defines a unique continuous field of C*-algebras $\mathfrak{F} = (A_t, \theta)$ where $\theta \supset A$. As A is total in θ, x a vector field is continuous with respect to A iff x is continuous with respect to θ by Proposition 2.3.4. Let A' denote the C*-algebra defined by $\mathfrak{F} = (A_t, \theta)$. A is a separating sub C*-algebra of A' hence by Corollary 2.3.16, $A' = A$. As A' is equal to the set of all continuous vector fields with respect to A which vanish at infinity, A is maximal.

In the following two lemmas and theorem A will denote a full algebra of vector fields on a locally compact Hausdorff space T with component C*-algebras A_t.

3.2.5. Lemma. Let $s \in T$ and $\pi_1, \ldots, \pi_n$ be a finite number of pairwise orthogonal non-zero projections in A_s. Then there exists a neighbourhood U of s and n elements $p_1, \ldots, p_n$ of A such that:

i) $p_i(s) = \pi_i$, $(i = 1, \ldots, n)$.

ii) $\forall\, t \in U$, $p_1(t), \ldots, p_n(t)$ are non-zero, pair-wise orthogonal projections in A_t.

Proof: Select n distinct positive integers $w_1, \ldots, w_n$. Define $B = \sum_{j=1}^{n} w_j \pi_j$ then B is a positive element belonging to

A_s . Choose a positive element $z \varepsilon A$ such that $z(s) = B$ and choose an ϵ such that $\frac{1}{4} \geqslant \epsilon > 0$. We shall now show that:

a) $\exists$ a neighbourhood U of s such that $\forall\, t \varepsilon U$, $z(t)$ has at least one eigenvalue in each interval $[\, w_j - \epsilon \,,\, w_j + \epsilon \,]$, no eigenvalue lying outside $[\, 0 \,,\, \epsilon \,]$ and also outside all $[\, w_j - \epsilon \,,\, w_j + \epsilon \,]$.

Fix j . Let F be a continuous non-negative function on the reals which is 0 outside $[\, w_j - \epsilon \,,\, w_j + \epsilon \,]$ and 1 at w_j . Then $(F(z))(s) = F(z(s)) = F(B) \neq 0$. Therefore $\| F(B)(s) \| \neq 0$ hence $\exists$ a neighbourhood U of s such that $\forall\, t \varepsilon U$, $\| F(z)(t) \| > 0 \Rightarrow F(z)(t) \neq 0 \Rightarrow z(t)$ has an eigenvalue in $[\, w_j - \epsilon \,,\, w_j + \epsilon \,]$.

Now choose a non-negative continuous function G on the reals which is 0 at 0 and at each w_j , $j = 1 , \ldots n$, and is 1 at all points which lie outside all the intervals $[\, -\epsilon, \epsilon \,]$ and $[\, w_j - \epsilon \,,\, w_j + \epsilon \,]$, $j = 1 , \ldots n$. Then $(G(z))(s) = G(z(s)) = G(B) = 0$. Therefore $\| G(z(s)) \| = 0$. Hence $\exists$ U a neighbourhood of s such that $\forall\, t \varepsilon U$, $\| G(z(t)) \| < 1$ and the first part of (a) holds. That is $z(t)$ has no eigenvalues at places where G is 1 . Hence (a) is satisfied.

For $j = 1 , \ldots n$ select a non-negative continuous function k_j on the reals such that k_j is 1 on $[w_j - \epsilon, w_j + \epsilon]$ and 0 outside $[w_j - 2\epsilon , w_j + 2\epsilon]$. If $P_j = k_j(z)$ then $P_j(s) = k_j(z(s)) = k_j(B) = \pi_j$. Hence (i).

If $t \varepsilon U$ then by (a) $P_j(t) = k_j(z(t))$ is a non-zero projection in A_t such that all the $P_j(t)$ are pair-wise orthogonal projections.

3.2.6. Lemma. Suppose that $s \varepsilon T$, π_1 and π_2 are projections in A_s , and α is an element of A_s such that $\alpha^*\alpha = \pi_1$, $\alpha\alpha^* = \pi_2$. Suppose further that p_1 and p_2 are elements of A such that:

i) $p_i(s) = \pi_i (i = 1, 2)$; and
ii) there is a neighbourhood U of s such that $p_1(t)$ and $p_2(t)$ are projections for all t in U .

Then there is an element q in A , and a neighbourhood V of s , such that $q(s) = \alpha$ and $(q^*q)(t) = p_1(t)$, $(qq^*)(t) = p_2(t)$ for t in V .

Proof: Choosing an element h' in A such that $h'(s) = \alpha$, and setting $h = p_2h'p_1$, we have

$$h(s) = \pi_2\alpha\pi_1 = \alpha , \qquad (1)$$

and for t in U ,

$$(p_2h)(t) = (hp_1)(t) = h(t) . \qquad (2)$$

Now consider the positive square root $g = (h^*h)^{\frac{1}{2}}$. We have by (1)

$$g(s) = (h(s))^*h(s))^{\frac{1}{2}} = (\omega^*\omega)^{\frac{1}{2}} = \pi_1 = p_1(s) . \tag{3}$$

For t in U , by (2)

$$(h^*h)(t) = p_1(t)(h^*h)(t)p_1(t) ;$$

hence, since $p_1(t)$ is a projection,

$$g(t) = p_1(t)\, g(t)p_1(t) \quad (t \in U) . \tag{4}$$

Now by (3) $(g - p_1)(s) = 0$. Hence, narrowing U if necessary we may assume that

$$\| (g - p_1)(t) \| < \frac{1}{8} \; (t \in U) . \tag{5}$$

Let φ be a continuous non-negative-valued function on the reals such that $\varphi(0) = 0$ and $\varphi(r) = \frac{1}{r}$ if $| r - 1 | \leqslant \frac{1}{8}$. Forming the element $q = h.\varphi(g)$, we have for t in U

$$(q(t))^*q(t) = \varphi(g(t))h(t)^*h(t)\varphi(g(t)) = (\varphi(g(t))g(t))^2 = (\psi(g(t)))^2 , \tag{6}$$

where

$$\psi(r) = r\varphi(r) = \begin{matrix} 0 & \text{if} & r = 0 , \\ 1 & \text{if} & | r - 1 | \leqslant \frac{1}{8} . \end{matrix} \tag{7}$$

Combining (4), (5), and (7), we find that $\psi(g(t)) = p\,(t)$ for t in U ; so, by (6),

$$(q(t))^*q(t) = p_1(t) \quad (t \in U) . \tag{8}$$

Now $\qquad q(s) = h(s)\varphi(g(s)) = \alpha\pi_1 = \alpha$; $\qquad$ (9)

so $\quad (qq^*)(s) = \alpha\alpha^* = \pi_2 = p_2(s)$, that is,

$$(qq^* - p\ )(s) = 0 \ . \tag{10}$$

On the other hand, for t in U , by (2)

$$p_2(t)(qq^*)(t) = p_2(t)h(t)\varphi(g(t))q^*(t) = h(t)\varphi(g(t))q^*(t) = (qq^*)(t) \ . \tag{11}$$

In view of (8), $q(t)$ is a partial isometry for t in U; thus, $(qq^*)(t)$ is a projection, which by (11) is contained in $p_2(t)$. If $(qq^*)(t) \neq p_2(t)$ for some t in U , then $\| (p_2 - qq^*)(t) \| = 1$. By (10) and the continuity of $\| (p_2 - qq^*)(t) \|$, the neighbourhood U can be narrowed so that for t in U this cannot happen. Then, for all t in U

$$(q(t))^*q(t) = p_1(t) \ , \quad q(t)(q(t))^* = p_2(t) \ ;$$

and this with (9) completes the proof.

3.2.7. Theorem. Suppose that $s \in T$, and that B is a finite dimensional*-subalgebra of A_s . Then there is a neighbourhood U of s , and a mapping $\beta \to x_\beta$ of B into A , such that

i) $\quad x_\beta(s) = \beta$ for all β in B ;

ii) $\quad$ for each t in U , $\beta \to x_\beta(t)$ is a *-isomorphism of B onto a finite dimensional*-subalgebra of A_t .

Proof: Let $B^1, \ldots, B^r$ be the minimal two-sided ideals of B ; and let $\{\beta^i_{jk}\}j,k = 1 , \ldots , n_i$ form a basis of B^i , where

$$\beta^{i}_{jk}\beta^{i}_{pq} = \delta_{kp}\beta^{i}_{jq}, \ (\beta^{i}_{jk})^* = \beta^{i}_{kj} .$$

Using Lemmas 3.2.6 and 3.2.5 we choose a neighbourhood U of s, and elements p^i_j $(i = 1, .., r ; j = 1, .., n_i)$ and $q^i_{j1}(i = 1, .., r ; j = 2, .., n_i)$ of A such that

i) $p^i_j(s) = \beta^i_{jj}$, $q^i_{j1}(s) = \beta^i_{j1}$;

ii) for each t in U, the $p^i_j(t)$ are $\sum_{i=1}^{r} n_i$ orthogonal non-zero projections;

iii) for each t in U, $i = 1, .., r$, and $j = 2, .., n_i$

$$(q^i_{j1}(t))^* q^i_{j1}(t) = p^i_1(t) ,$$
$$q^i_{j1}(t)(q^i_{j1}(t))^* = p^i_j(t) .$$

Now define $q^i_{jk} = q^i_{j1}(q^i_{k1})^*$. We easily verify that the linear map of B into A which carries β^i_{jk} into q^i_{jk} has the required properties.

3.2.8. Definition: A C*-algebra A is homogeneous of dimension n if every irreducible representation of A is of the same finite dimension n.

We shall now show how to represent a homogeneous C*-algebra as the set of all continuous cross-sections vanishing at infinity of a fibre bundle $\mathfrak{B}$. First of all the definition of a fibre bundle will be given.

3.2.9. <u>Definition</u>:

<u>A co-ordinate bundle</u> $\mathfrak{B}$ is a collection as follows:

i) a topological space B called the bundle space,

ii) a topological space X called the base space,

iii) a continuous map $p : B \to X$ of B onto X called the projection map,

iv) a space Y called the fibre and the set $Y_x = p^{-1}(x)$ called the fibre over the point x of X such that each Y_x is homeomorphic to Y,

v) an effective topological transformation group G of Y called the group of the bundle. That is G is a topological group and Y a topological space such that $\eta : G \times Y \to Y$ is continuous and $\eta(e,y) = y$ where e is the identity of G and $\eta(g_1 g_2, y) = \eta(g_1, \eta(g_2, y))$ $\forall\, g_1, g_2 \in G$, $y \in Y$. G is said to be effective if $\eta(g,y) = y \;\; \forall\, y \Rightarrow g = e$. Then G is isomorphic to a group of homeomorphisms of Y.

vi) a family $\{V_j\}$ of open sets covering X indexed by a set J, the V_j are called co-ordinate neighbourhoods and

vii) $\forall\, j \in J$, a homeomorphism $\varphi_j : V_j \times Y \to p^{-1}(V_j)$ called the co-ordinate function.

viii) $p\varphi_j(x,y) = x \quad \forall\, x \in V_j, \;\; y \in Y$

ix) if the map $\varphi_{j,x} : Y \to p^{-1}(x)$ is defined by setting $\varphi_{j,x}(y) = \varphi_j(x,y)$, then for each pair i, j in J and each $x \in V_i \cap V_j$, the homeomorphism $\varphi_{j,x}^{-1}\varphi_{i,x} : Y \to Y$ coincides with the operation of an element of G (it is unique since G is effective) and

x) for each pair i, j in J, the map $g_{ji} : V_i \cap V_j \to G$ defined by $g_{ji}(x) = \varphi_{j,x}^{-1}\varphi_{i,x}$ is continuous. The functions g_{ji} are called the co-ordinate transformations of the bundle.

3.2.10. Definition :

Two co-ordinate bundles $\mathfrak{B}$ and $\mathfrak{B}'$ are said to be equivalent in the strict sense if they have the same bundle space, base space, projection, fibre and group and their co-ordinate functions $\{\varphi_j\}$, $\{\varphi_k'\}$ satisfy the conditions that

$$\bar{g}_{kj}(x) = \varphi_{k,x}'^{-1}\varphi_{j,x} \qquad x \in V_j \cap V_k'$$

coincides with the operation of an element of G and the map $\bar{g}_{k,j} : V_j \cap V_k' \to G$ so obtained is continuous. This is an equivalence relation, that is it is reflexive, symmetric (because the map $g \to g^{-1}$ is continuous) and transitive, (because the map $(g_1, g_2) = g_1 g_2$ is continuous).

A fibre bundle is defined to be an equivalence class of co-ordinate bundles.

This next theorem will show how an n dimensional homogeneous C*-algebra can be constructed from a fibre bundle.

Let M_n be the C*-algebra of all n by n matrices over $\mathbb{C}$. G_n the group of all automophisms of M_n of the form $a \to u^{-1}au$ where u is a unitary matrix in M_n . G_n becomes a topological transformation group of M_n in the simple convergence topology over M_n . For $\mathfrak{B}$ a fibre bundle over a locally compact Hausdorff base space T , then $C_o(\mathfrak{B})$ denotes the family of all continuous cross sections of $\mathfrak{B}$ vanishing at ∞ . That is x is a continuous map from the base space into the bundle space so that $p(x(t)) = t \quad \forall\, t$ in the base space and $\lim_{t\to\infty} \| x(t) \| = 0$.

3.2.11. Theorem. Let $\mathfrak{B}$ be a fibre bundle with bundle space B , base space T which is locally compact Hausdorff, fibre space M_n and group G_n then $C_o(\mathfrak{B})$ is a n-dimensional homogeneous C*-algebra, such that $C_o(\mathfrak{B})^\wedge$ is homeomorphic with T .

Proof: If p is the projection of B onto T , $A_t = p^{-1}(t)$ $(t \in T)$, $\{V_j\}$ a covering of T by co-ordinate neighbourhoods and $\{\varphi_j\}$ the corresponding co-ordinate functions then we can transfer to each fibre A_t the algebraic operations and the C*-algebraic norm of M_n via the mapping $\varphi_{j,t}(\alpha) = \varphi_j(t,\alpha)$ where j is chosen such that $t \in V$. This makes each A_t into a C*-algebra isometrically-*-isomorphic with M_n . The operations in A_t thus defined are independent of the choice of j . $C_0(\mathfrak{P})$ is a C*-algebra under the pointwise operations and supremum norm. In fact it is a maximal full algebra of vector fields with component algebras A_t . Thus by Theorem 3.2.3 $C_0(\mathfrak{P})$ is homeomorphic with T . Hence $C_0(\mathfrak{P})$ is homogeneous of dimension n .

Using the Stone-Weierstrass theorem we shall now prove the representation theorem for homogeneous C*-algebras.

3.2.12. Theorem. Every homogeneous C*-algebra A of dimension n is isometrically-*-isomorphic with some $C_0(\mathfrak{P})$, where $\mathfrak{P}$ is a fibre bundle with base space $\hat{A}$, fibre space M_n , group G_n .

Proof: $\hat{A}$ is locally compact and Hausdorff by Theorem 2.1.13. Every $a \in A$ can be considered to be a function $a' : \hat{A} \to B$ defined by $a'(T) = (T, T(a))$ where $B = \{ (T,\alpha) ; T \in \hat{A}$ and $\alpha \in T(A) \}$. Set $A' = \{ a' ; a \in A \}$. Then A is isometrically-*-isomorphic to A' which is a maximal full algebra of vector fields by Theorem 3.2.4, Theorem 2.1.12 and Proposition 2.1.11. We shall construct a fibre bundle with base space B . For this we choose

i) A covering of $\hat{A}$ by open sets $\{ U_i \}$

ii) For each i a mapping f_i of M_n onto A , such that wherever $T \in U_i$ the mapping $a \to T_{f_i(a)}$ is a

*-isomorphism of M_n onto $T(A)$. This is possible by Theorem 3.2.7.

Define $\varphi_i : U_i \times M_n \to \{ (t, a) ; T \in U_i$ and $a \in T(A) \}$ by $\varphi_i(t, a) = (T, T_{f_i(a)})$ then φ_i is one to one onto. Let us consider the topology on $p^{-1}(U_i)$ induced by the map φ_i from $U_i \times M_n \to p^{-1}(U_i)$. This induces the unique topology on $p^{-1}(U_i \cap U_j)$. Therefore $\{ \varphi_i \}$ defines a topology on B and we shall now consider the space B endowed with this topology.

Let $\mathfrak{B}$ denote the fibre bundle with base space $\hat{A}$, bundle space B , projection p , group G_m , fibre M_m , co-ordinate neighbourhoods $\{ U_i \}$ and co-ordinate functions $\{ \varphi_i \}$.

i) p is a continuous map from B onto $\hat{A}$.

ii) φ_i is a homeomorphism from $U_i \times M_n \to p^{-1}(U_i)$

iii) The co-ordinate transformations $g_{ji} : U_i \cap U_j \to G_n$ defined by $g_{ji}(T) = \varphi_{j,T}^{-1} \varphi_{i,T}$ must be continuous . This amounts to saying that for each a in M_n , the map $a \to g_T(a)$ is continuous on $U_i \cap U_j$, where $g_T(a) = b$ is the element of M_n defined by $T_{f_j(a)} = T_{f_i(b)}$. This follows from the continuity of the function $T \to \| T_x \|$, $\forall x \in A$.

A vector field x on A is continuous with respect to A iff the map $T \to (T , x(T))$ is a continuous cross section of $\mathfrak{B}$ hence the family $C_o(\mathfrak{B})$ of all continuous cross sections of $\mathfrak{B}$ vanishing at infinity coincides with the maximal full algebra of vector fields A'. That is $C_o(\mathfrak{B}) = A'$ hence A is isometrically-*-isomorphic to $C_o(\mathfrak{B})$.

Having represented A as a set of functions A' on $\mathring{A}$ we shall now see under which conditions the spectrum of an element $a \in A$ is contained in its image under this representation.

In the case that A is a commutative C*-algebra with identity, we have that A is isometrically-*-isomorphic to $C(\Phi_A)$, (See Theorem 1.3.6.) and $\forall\ a \in A$, $Sp_A(a) = \{a'(\varphi); \varphi \in \Phi_A\}$ where a' is the representation of a in $C(\Phi_A)$.

Let A be a normed algebra over $\mathbb{R}$ or $\mathbb{C}$ with an identity. Define $V_A(a) = \{f(a)\ ;\ f$ is a continuous linear functional on A and $\| f \| = f(e) = 1\}$, $\forall\ a \in A$.

3.2.13. Theorem. A a complex Banach algebra with identity, then $\forall\ a \in A$, $Sp_A(a) \subset V_A(a)$.

Proof: Let $\lambda \in Sp_A(a)$ then $\lambda e - a$ has no inverse in A. Suppose that it has no right inverse, then $(\lambda e - a)A = J$ is a proper right ideal of A. Since A is complete $\| x - e \| \geq 1$ $\forall\ x \in J$. Hence by the Hahn-Banach theorem $\exists$ a continuous linear functional f defined on A such that $f(e) = \| f \| = 1$ and $f(J) = 0$. Hence $f(\lambda e - a) = 0 \Rightarrow \lambda = f(a) \in V_A(a)$.

3.2.14. Theorem. A a complex C*-algebra with an identity. If c is a normal element of A then $Sp_A(c) \subseteq \{f(c) \; ; \; f \in P(A)\}$.

See C.E. Rickart [29].

3.2.15. Proposition. A an n-dimensional homogeneous C*-algebra with identity, then

i) $\forall \, a \in A$, $Sp_A(a) \subseteq \{(T_a\xi,\xi) \; ; \; T$ is a representation of A and $\xi \in H_T$ is such that $(\xi,\xi) = 1\}$.

ii) For c a normal element belonging to A then $Sp_A(c) \subseteq \{(c'(T)\xi,\xi) \; ; \; T \in \hat{A}, \; \xi \in H_T$ with $(\xi,\xi) = 1\}$.

iii) If $n = 1$ then for c a normal element belonging to A $Sp_A(c) \subseteq \{(c'(T)\xi,\xi) \; ; \; T \in \hat{A}$ and ξ belongs to the orthonormal base of $H_T\}$.

Proof: (i) By Theorem 3.2.13. $Sp_A(a) \subset V_A(a)$ $\forall \, a \in A$, but by Proposition 1.3.9. a continuous linear functional f on a complex B*-algebra such that $\| f \| = f(e) = 1$ is necessarily positive. Hence $V_A(a) = \{f(a) \; ; \; f \in S(A)\}$. But $\forall \, f \in S(A)$ $\exists$ a representation T of A by Proposition 1.3.11. such that $f(x) = (T_x\xi,\xi)$, $\xi \in H_T$ and $\| f \| = (\xi,\xi) = 1$. Therefore $\{f(a) \; ; \; f \in S(A)\} \subseteq \{(T_a\xi,\xi) \; ; \; T$ is a representation of A, $\xi \in H_T$ and $(\xi,\xi) = 1\}$.

(ii) By Theorem 3.2.14, $Sp_A(c) \subseteq \{f(c) \; ; \; f \in P(A)\}$. By Proposition 1.3.11, $\forall \, f \in P(A)$, $\exists \, T \in \hat{A}$ such that $f(x) = (T_x\xi,\xi)$ and $\| f \| = (\xi,\xi) = 1$, $\xi \in H_T$.

By Theorem 3.2.12. c can be considered to be a function $c' : A \to B$ such that $c'(T) = T_c$. Hence $\{f(a) ; f \varepsilon P(A)\} \subseteq \{(c'(T)\zeta,\zeta) ; T \varepsilon \hat{A}, \zeta \varepsilon H_T$ and $(\zeta,\zeta) = 1\}$.

(iii) If $n = 1$ then $\forall\, T \varepsilon \hat{A}$ the dimension of H_T is one. Hence $\forall\, \zeta \varepsilon H_T$ such that $(\zeta,\zeta) = 1$, $\exists\, \lambda \varepsilon \mathbb{C}$ such that $|\lambda| = 1$ and $\zeta = \lambda\eta$ where η belongs to the orthonormal base of H_T . Therefore by Proposition 1.3.12 $(T_x\zeta,\zeta) = (T_x\eta, \eta)$, $\forall\, x \varepsilon A$. Therefore $\{(c'(T)\zeta,\zeta) ; T \varepsilon \hat{A}, \zeta \varepsilon H_T$ and $(\zeta,\zeta) = 1\} =$ $\{(c'(T)\zeta,\zeta) ;$ $T \varepsilon \hat{A}$ and ζ belongs to the orthonormal base of $H_T\}$.

Section 3. The Topology Defined by a Continuous Field of Banach Spaces.

Let $\mathfrak{J} = (A_t, \theta)$ be a continuous field of Banach spaces on a topological space T . Let $B = \bigcup_{t\varepsilon T} A_t$ and p be the natural map from $B \to T$ defined by $p(x) = t$ when $x \varepsilon A$. We will identify A_t and $p^{-1}(t)$. $\forall$ open subset Y of T $\epsilon > 0$ and $x \varepsilon \theta$ let $T(Y, x, \epsilon) = \{\zeta \varepsilon B ; p(\zeta) \varepsilon Y$ and $\| \zeta - x(p(\zeta)) \| < \epsilon\}$ then $\mathfrak{B} = \{T(Y, x, \epsilon) ;$ Y open in T ,

$\epsilon > 0$, $x \varepsilon \theta \}$ forms a basis for a topology in B such that

i) the topology induced on the A_t coincides with the norm topology on the A_t .

ii) x a vector field is continuous with respect to iff x is a continuous map from $T \to B$ such that $p_\circ x = I_T$.

iii) $B \nabla B = \{(b, b') \varepsilon B \times B ; p(b) = p(b')\}$. Then the mapping $(b, b') \to b + b'$ from $B \nabla B \to B$ is continuous, the mapping $(\lambda, b) \to b$ from $\mathbb{C} \times B \to B$ is continuous and the mapping $b \to \|b\|$ from $B \to \mathbb{R}$ is continuous.

See J. Dixmier and A. Douady [7].

J. Tomiyama [31] considers the question whether it is always possible to define a natural topology in the set $B = \bigcup_{P \varepsilon Prim(A)} A/P$ where A is a C*-algebra so that A may be represented as the algebra of all continuous sections defined on Prim(A) into B even though Prim(A) is not necessarily Hausdorff. He shows that if A is a C*-algebra such that there exists an appropriate decomposition of Prim(A) then we get a locally compact Hausdorff space X at each point of which a C*-algebra A is given and setting $B = \bigcup_{x \varepsilon X} A$ and giving B the above defined topology, A can be represented as the algebra of all continuous sections vanishing at infinity defined on X .

This result will be proved for an arbitary C*-algebra in the next chapter using the concept of a uniform field.

Notes and Remarks.

For the properties of biregular rings see N. Jacobson [15] and R. Arens and I. Kaplansky [1]. The representation theorem (Theorem 3.1.11) and Theorem 3.1.12. are due to J. Dauns and K.H. Hofmann [4]. Theorem 3.1.11 has been generalized by J. Dauns and K.H. Hofmann [5] to the following case. "A a weakly biregular ring (that is for an ordered pair (I_1, I_2) of maximal modular ideals $\exists$ e a central idempotent such that $e \in I_1$ and $e \notin I_2$ and $\bigcap_{x \in M(A)} x = 0$ where $M(A)$ denotes the space of maximal modular ideals in the hull kernel topology) such that every proper ideal is contained in some maximal modular ideal, then A is isomorphic to $\Gamma_0(\pi)$ where $\pi : E \to M(A)$ is a sheaf of rings whose stalks are local rings with identy."

Theorems 3.2.3. and 3.2.4. were obtained from the results of J. Dixmier [6]. Theorem 3.2.7, the representation theorem (Theorem 3.2.12.) and its converse Theorem 3.2.11. are due to J.M.G. Fell [9]. J. Tomiyama and M. Takesaki [33] proved a result similar to Theorem 3.2.12, using Prim(A) as the base space which in this case is homeomorphic to $\hat{A}$. Theorem 3.2.13. is due to F. Bonsall [2] and Proposition 3.2.15. was constructed using the results of J. Dixmier [6], C.E. Rickart [29] and F. Bonsall [2].

CHAPTER IV

The Representation of an Arbitrary Complex B*-Algebra.

In this chapter we shall prove a representation theorem for an arbitrary C*-algebra. (Theorem 4.3.6.) In the previous chapters the type of ring considered always had a Hausdorff structure space and hence we used the structure space as the base space. Unfortunately in general the structure space is not Hausdorff let alone completely regular and as we shall be needing a general type of Stone-Weierstrass theorem we require the base space to be completely regular. Hence we shall not be representing the algebra over the structure space, but over a completely regular space which is associated with any topological space.

In Section I the concept of a uniform field will be introduced and Theorem 4.1.4. proves the existence of a uniform field given a topological group and a family of normal subgroups. A uniform field is a generalization of a sheaf and a fibre bundle and has the advantage of having non-discrete stalks which vary continuously with their dependence on the base point. In Section 2 the type of Stone-Weierstrass theorem which will be used is given, (Theorem 4.2.4.) and we shall then use it to prove the representation theorem, (Theorem 4.3.2.). Finally in Section 3, the generalised Gelfand-Naimark theorem will be proved using the representation theorem. It shows us that any C*-algebra is isometrically-*-isomorphic to the C*-algebra of continuous sections vanishing at infinity in a uniform field with a completely regular base space, (Theorem 4.3.6.)

Section 1. A Uniform Field.

4.1.1. Definition. Let E , B be topological spaces and $\pi : E \to B$ a surjective continuous function. B will be called the base space, E the field space and the subspaces $\pi^{-1}(b)$, $b \in B$ the stalks of π . A section is a continuous function $\sigma : V \to E$ where V is open in B , such that $\pi \circ \sigma = I_V$, the identity on V. If $V = B$ then σ is called a global section. $\Gamma(\pi)$ denotes the set of global sections and the set of sections with domain V (an open set in B) is denoted by $\Gamma(V, \pi)$. The map $\pi : E \to B$ is called a field of topological spaces if $E = \cup \{\sigma(V) ; V$ open in B , $\sigma \in \Gamma(V,\pi)\}$. $E \nabla E = \{(x, y) \in E \times E; \pi(x) = \pi(y)\}$.

A field uniformity $\mathfrak{U}$ for π is a filter on $E \nabla E$ such that each $U \in \mathfrak{U}$ contains $\Delta E = \{(x, x) ; x \in E\}$ and that $\{U \circ U^{-1} ; U \in \mathfrak{U}\}$ is a basis for $\mathfrak{U}$. For each section σ and each $U \in \mathfrak{U}$, $U(\sigma) = \{ x \in E ; \pi(x) \in$ domain σ and $(\sigma(\pi(x)), x) \in U\}$. Now if the neighbourhood filter of each point $x \in E$ has as a basis the set of sets $U(\sigma)$ with $U \in \mathfrak{U}$ and sections σ such that $\pi(x) \in$ domain σ and $x = \sigma(\pi(x))$ then the pair $(\pi, \mathfrak{U})$ is called a <u>uniform field</u>.

Hence for a uniform field the uniform topologies on the stalks agree with the topology induced from the given topology on E.

If for a uniform field $(\pi, \mathfrak{U})$ the set $\Gamma(V,\pi)$ is non-empty then the set $\tilde{U} = \{(\sigma,\tau) \in \Gamma(V,\pi)\times\Gamma(V,\pi)\ ;\ (\sigma(b),\tau(b)) \in U,\ \forall\, b \in V\}$, $U \in \mathfrak{U}$ is the basis of a uniform structure on $\Gamma(V,\pi)$.

We shall now answer the question about the existence of uniform fields given $\pi : E \to B$ a surjective function of sets, Σ a full set of global cross sections that is

$\Sigma = \{\sigma; \sigma : B \to E\}$, $\pi_\circ\sigma = I_B, \forall\, \sigma \in \Sigma$ and $E = \cup\,\{\text{image } \sigma\ ;\ \sigma \in \Sigma\}$ and $\mathfrak{U}$ a field uniformity for π.

4.1.2. <u>Definition</u>. The <u>quasi-interiors</u> U° of $U \in \mathfrak{U}$ is the set of all pairs $(x,y) \in U$ which satisfy the following two symmetric conditions

a) $\forall\, \alpha \in \Sigma$ with $\alpha(\pi(x)) = x$, $\exists\, V \in \mathfrak{U}$ and $\beta \in \Sigma$ such that $V(\beta) \subset U(\alpha)$ and $\beta(\pi(y)) = y$

b) $\forall\, \beta \in \Sigma$ with $\beta(\pi(y)) = y$, $\exists\, V \in \mathfrak{U}$ and $\alpha \in \Sigma$ such that $V(\alpha) \subset U(\beta)$ and $\alpha(\pi(x)) = x$.

If $U = U^\circ$ then U is said to be quasi-open.

If $\{U^\circ ; U \in \mathfrak{U}\}$ is a basis of $\mathfrak{U}$ then $\{U^\circ(\sigma) ; U \in \mathfrak{U} , \sigma \in \Sigma\}$ is a sub-basis for a topology $N(\Sigma)$ on E. The finest topology on B making π a continuous map from $(E , N(\Sigma)) \to B$ will be denoted by Q. Let $T(\Sigma)$ denote the coarsest topology on B making all sections in Σ continuous and it is called the weak-star topology. The collection of sets $W \cap \pi^{-1}(V)$, $W \in N(\Sigma)$, $V \in T(\Sigma)$ is a basis for a topology $N(\pi)$ on E and $(\pi, \mathfrak{U})$, $\pi : (E, N(\pi)) \to (B, T(\Sigma))$ is a uniform field such that $\Sigma \subset \Gamma(\pi)$.

We shall now show that given any topological group G together with an indexed family $\{G_b ; b \in B\}$ of normal subgroups of G, a uniform field $\pi : E = \cup \{ G/G_b ; b \in B\} \to B$ is constructed such that G has a homomorphic image $\hat{G}$ in the group $\Gamma(\pi)$ of all continuous sections of the field.

4.1.3. Definition. A field of topological semigroups is a field of topological spaces $\pi : E \to B$ all of whose stalks $\pi^{-1}(b)$, $b \in B$ are semigroups such that the function $(x, y) \to xy : E \nabla E \to E$ is continuous. A field of topological groups is a field of topological semigroups all of whose stalks are groups such that the function $x \to x^{-1} : E \to E$ is continuous. A field of topological rings is a field whose stalks are rings which with respect to addition is a field of topological groups and a field of topological semigroups with respect to multiplication. It is a field of topological rings with identity if each stalk $\pi^{-1}(b)$, $b \in B$ contains a multiplicative identity $1(b)$, and if the map $b \to 1(b) : B \to E$ belongs to $\Gamma(\pi)$. A left (respectively right) uniform field of topological groups is at the same time both a uniform field of topological spaces $(\pi, \mathfrak{U})$, $\pi : E \to B$ and also a field of topological groups such that $\{U \cap \pi^{-1}(b)\times\pi^{-1}(b) \; ; \; U \in \mathfrak{U}\}$ is the left (respectively right) uniform structure of the group $\pi^{-1}(b)$ for each $b \in B$ and $\forall\, U \in \mathfrak{U}$, $\exists\, W \in \mathfrak{U}$ such that (x, s), $(y, t) \in W$ and $\pi(x) = \pi(y) \Rightarrow (x^{-1}y, s^{-1}t) \in U$ (respectively $(yx^{-1}, ts^{-1}) \in U$).

If the groups are abelian then π is called a uniform field of abelian groups. A field of topological rings is called a uniform field of topological rings if it is a uniform field of abelian groups relative to the additive structure.

Let $(\pi, \mathfrak{U})$, $\pi : E \to B$ be a left uniform field of groups. For each open subset $V \subset B$ and for $\sigma, \tau \in \Gamma(V)$ set $(\sigma^{-1}\tau)(v) = \sigma(v)^{-1}\tau(v) \quad \forall\, v \in V$. Then the operation $(\sigma, \tau) \to \sigma^{-1}\tau : \Gamma(V)\times\Gamma(V) \to \Gamma(V)$ defines on $\Gamma(V)$ the structure of a topological group whose left uniform structure coincides with the natural uniformity of $\Gamma(\pi)$. We now have the existence theorem for a topological group G.

4.14. Theorem. For any topological group G having a neighbourhood basis $\mathfrak{B}$ of the identity invariant under all inner automorphisms and any family $\{G_b \; ; \; b \in B\}$ of normal subgroups let E be the disjoint union of the sets G/G_b , $b \in B$ and define $\pi : E \to B$ by $\pi(G_b g) = b$ and let $\hat{g}$ denote the Gelfand transformation of $g \in G$ where $\hat{g} : B \to E$ defined by $\hat{g}(b) = G_b g$. Let $\mathfrak{U}$ be the filter on $E \nabla E$ generated by all the sets $U' = \{(x, y) \in E \nabla E \; ; \; y \in x U\}$ where $U \in \mathfrak{B}$. Then

i) $(\pi, \mathfrak{U}) : (E, N(\hat{G})) \to (B, T(\hat{G}))$ is a left uniform field of topological groups. The topologies $N(\hat{G})$, $T(\hat{G})$ are the unique weakest ones for which $(\pi, \mathfrak{U})$ is a uniform field with $\hat{G} \subset \Gamma(\pi)$.

ii) the map $(\cap B)g \to \hat{g} : G/\cap B \to \hat{G} \subset \Gamma(\pi)$ is a uniformly continuous injective morphism of topological groups.

iii) $\{U'(\hat{g}) \; ; \; U \in \mathfrak{B} , g \in G\}$ is a sub-basis for $N(\hat{G})$.

iv) $\{U(g) \; ; \; U \in \mathfrak{B} , g \in G\}$ where $U(g) = \{b \in B \; ; \; g \in G_b \; U\}$ is a sub-basis for $T(\hat{G})$.

See J. Dauns and K.H. Hofmann [5] .

Section 2. A Stone-Weierstrass Theorem in a Uniform Field.

We shall now define a concept of boundedness for $\Gamma(\pi)$ and a notion of vanishing at infinity which generalises the notion of vanishing at infinity in a normed space.

4.2.1. Definition. Let $(\pi, \mathfrak{U})$, $\pi : E \to B$ be a left uniform field of groups and $\zeta(b)$ denote the identity of the group $\pi^{-1}(b)$, $\forall\, b \in B$. Let $\Gamma^b(\pi) = \{\tau \in \Gamma(\pi) ;\ \forall\, U \in \mathfrak{U}$ where $U = U^{-1}$, $\exists\, n \in \mathbb{N}$ such that $(\tau(b), \zeta(b)) \in U^n$, $\forall\, b \in B\}$. Then $\Gamma^b(\pi)$ is a closed normal subgroup of $\Gamma(\pi)$. If B is completely regular and $\mathfrak{X}$ a set of compact subsets of B closed under finite unions then a section $\sigma \in \Gamma(\pi)$ is said to *vanish at infinity* if $\forall\, U \in \mathfrak{U}$, $\exists\, C \in \mathfrak{X}$ such that $(\zeta(b), \sigma(b)) \in U$, $\forall\, b \in B \sim C$. The subgroup of $\Gamma^b(\pi)$ of all sections vanishing at infinity will be denoted by $\Gamma_0(\pi, \mathfrak{X})$ or just $\Gamma_0(\pi)$ if $\mathfrak{X}$ is fixed. If $B \in \mathfrak{X}$ then $\Gamma_0(\pi)$ can be identified with $\Gamma^b(\pi)$.

By a *topological ring* R will be meant a ring R with a Hausdorff topology making addition and multiplication as maps $R \times R \to R$ continuous. If A and K are topological rings and K is commutative then A is a *topological algebra* over K provided A is an algebra over K and multiplication $K \times A \to A$ is continuous.

4.2.2. *Definition*. A uniform field $(\pi, \mathfrak{U})$, $\pi : E \to B$ of real vector spaces is called *locally convex* provided $\mathfrak{U}$ has a basis of elements U such that $U(\theta) \cap \pi^{-1}(b)$ (where θ is the identity section) is convex $\forall\, b \in B$. A field of complex vector spaces is called locally convex if it is locally convex when scalar multiplication is restricted to the reals.

Frequently an algebra A over a field K can be faithfully represented as a subset of the set of all sections in a field $\pi : E \to B$. A partition of unity is used firstly to show that A is a module over the ring $C^b(B, K)$ of all bounded continuous functions of B into K and secondly to show that A is isomorphic to $\Gamma^b(\pi)$ or $\Gamma_o(\pi)$.

4.2.3. Definition. Let $(\pi, \mathfrak{U})$, $\pi : E \to B$ be a uniform field of rings and F denote the stalkwise ring multiplication. A set $X \subset E$ is said to be multiplicatively bounded if for any $U \in \mathfrak{U}$, $\exists\, V \in \mathfrak{U}$ such that $F[X \nabla V(0)] \subset U(0)$. Suppose that $S \subset \Gamma(\pi)$ is a subring and for any $\sigma_1, \sigma_2 \in S$ the map $\overline{F}[\sigma_1, \sigma_2] : B \to E$ is defined by $\overline{F}[\sigma_1, \sigma_2]\,(b) = F(\sigma_1(b), \sigma_2(b))$. Then if C is a locally finite open cover of B and $U \in \mathfrak{U}$, $\sigma \in S$ then a partition of unity in S relative to σ up to U subordinate to C is a function $W \to e_W : C \to S$ such that

i) $\overline{F}[e_W, \sigma] \mid (B\sim W) = 0(B\sim W)$,

ii) $\Sigma\{\overline{F}[e_W, \sigma] ; W \in C\} \in \tilde{U}(\sigma)$.

S is said to have bounded approximate partitions of unity relative to itself and a set of locally finite open covers of B if $\forall\, U \in \mathfrak{U}$ and $\sigma \in S$ and each locally finite open cover C there is a partition of unity $\{e_W ; W \in C\}$ in S relative to σ up to U such that $e_W(B) \subset X$, $\forall\, W \in C$ and for some fixed multiplicatively bounded set $X \subset E$.

We then have the following Stone-Weierstrass theorem which

which will be needed to prove the representation theorem in Section 3.

4.2.4. Theorem. Let $(\pi, \mathfrak{U})$, $\pi : E \to B$ be a locally convex uniform field of algebras over K, where K is either the reals or complexes. Suppose $\Gamma^b(\pi)$ is a $C^b(B, R)$ module. Let $A \subset \Gamma^b(\pi)$ be a subalgebra such that scalar multiplication $K \times A \to A$ is continuous and such that for any $U \in \mathfrak{U}$, $\exists$ a zero neighbourhood N in K and $V \in \mathfrak{U}$ with $\{(hx, ky) ; (x\ y) \in V , k - h \in N\} \subset U$. Assume that A satisfies the following conditions:

a) A has approximate bounded partitions of unity relative to itself and all finite open covers of B;

b) the closure of $A(b)$ in $\pi^{-1}(b) = \Gamma^b(\pi)(b)$, $\forall\, b \in B$ and that at least one of the following conditions holds:

α) B is compact Hausdorff.

β) B is completely regular and there is a family $\mathfrak{X}$ of compact subsets of B such that $A \subset \Gamma_o(\pi, \mathfrak{X})$ then

i) $\overline{A}$ is a $C^b(B, K)$ module;

ii) (α') $\overline{A} = \Gamma^b(\pi) = \Gamma(\pi)$ in case (α)

(β') $\overline{A} = \Gamma_o(\pi, \mathfrak{X})$ in case (β).

For the proof see [5]. The partition of unity enables the authors to show that $\overline{A}$ is a module over the ring $C(B,K)$ of all bounded continuous functions of B into K. Then using this fact and the assumption that the closure of $A(b) = \Gamma^b(\pi)(b)$ $\forall\, b \in B$, they obtained, that in the case B is compact T_2 $\overline{A} = \Gamma^b(\pi) = \Gamma(\pi)$, and in the case B is completely regular $\overline{A} = \Gamma_o(\pi, \mathfrak{X})$.

Section 3. Representations of Algebras.

We will first of all introduce the concepts of partitions of unity for a topological ring.

4.3.1. Definition. A subset X of a topological ring will be said to be multiplicatively bounded if for any neighbourhood U of zero there is another zero neighbourhood V with $XV \subset U$ and $VX \subset U$. A pair A, B consisting of a topological ring and a set B of ideals of A topologized in some manner is said to have approximate bounded partitions of unity if there is a fixex multiplicatively bounded set $X \subset A$ such that for any open cover $B = U_1 \cup .. \cup U_n$, a neighbourhood V of zero in A, and any $a \in A$, $\exists\ e_k \in \cap \{b ; b \in B \sim U_k\} \cap X$, such that $a - ae \in V$ and $a - ea \in V$, with $e = e_1 + \ldots + e_n$.

Let A be a ring and B a set of prime ideals with the hull kernel topology. The advantage of the hull kernel topology is that it is compact if A has an identity and is locally compact if A is a C*-algebra but unfortunately is rarely completely regular which would be desirable in view of the application of the Stone-Weierstrass theorem 4.2.4. Hence we use the fact that if X is a topological space, there exists a continuous map $\varphi : X \to X'$ onto a completely regular space such that all continuous maps from X into a completely regular

space factor through φ . This follows from the existence theorem of adjoint functors [25]. Hence if B is a set of prime ideals, h a continuous surjection mapping $B \to B'$ a completely regular space then $m_{b'} = \cap \varphi^{-1}(b')$ for $b' \varepsilon B'$ is an ideal and set $M = \{m_{b'} ; b' \varepsilon B'\}$. M is given the topology T that makes the function $b' \to m_{b'}$ a homeomorphism and M is then called the complete regularisation of B . $\exists$ a continuous map $\varphi : B \to M$ such that $m = \cap \varphi^{-1}(m)$, $\forall m \varepsilon M$. Let the canonical field A, B be denoted by $(\pi, \mathfrak{U})$, $\pi : (E, N(\hat{A})) \to (B, Q)$ where $Q^* = T(\hat{A})$, then associated with this change of base space $\varphi : B \to M$ is a surjective function $\pi' : E' \to M$, a field uniformity $\mathfrak{U}'$, and a full set of sections $\tilde{A}$ where $\tilde{a} : M \to E'$ defined by $\tilde{a}(m) = a \bmod m$. Then in order for $(\pi', \mathfrak{U}')$ to be a uniform field the quasi-interiors of the elements of $\mathfrak{U}'$ have to form a basis of $\mathfrak{U}'$ and the resulting uniform field will be denoted by
$(\pi', \mathfrak{U}')$, $\pi' : (E', N(\tilde{A})) \to (M, T^*)$ where $T^* = T(\tilde{A})$.
$T \nabla T^*$ denotes the smallest topology containing T and T^* .
Refinement of the base space in π' gives the new field
$\pi'' : E'' \to (M, T \nabla T^*)$. In order for $(M, T \nabla T^*)$ to be completely regular we must have $T \subset T \nabla T^*$ and this is so if $\forall a \varepsilon A$ and each open neighbourhood U of zero in A , the set $\{b \varepsilon B ; a \notin b + U\}$ is hull kernel compact. Then
$T^* \subset T$ hence $T = T \nabla T^*$.

We then have the following representation theorem which is proved using the Stone-Weierstrass theorem 4.2.4. and the existence theorem 4.1.4.

4.3.2. Theorem. Let A be a locally convex complete topological algebra over the reals or complexes such that multiplication and scalar multiplication are uniformly continuous on bounded sets and such that A has a neighbourhood basis of 0 consisting of bounded neighbourhoods and the following conditions are satisfied:

i) A has bounded approximate partitions of unity relative to a set of prime ideals B with the hull kernel topology;

ii) $\{ b \in B ; a \notin b + U \}$ is hull kernel compact $\forall a \in A$ and any neighbourhood U of zero in A.

iii) For every neighbourhood U of zero, there is a zero neighbourhood W in A for which $\cap \{b + W ; b \in B\} \subset U$.

Then there is a set M of ideals of A with a completely regular topology T and a continuous surjective map $\varphi : B \to M$ with the hull kernel topology on B where $m = \cap \varphi^{-1}(m) \quad \forall m \in M$. The map φ has the property that any other map of B into any completely regular space factors uniquely through φ.

Let $(\pi', \mathfrak{U}')$, $\pi' : E' \to (M, T^*)$ be the field obtained by first forming the canonical field of A and M and $(\pi'', \mathfrak{U}')$, $\pi'' : E'' \to (M, T)$ be the field obtained by refining the weak star topology T^* of M . $(T^* \subset T)$.

a) If $e \in A$ and B contains all maximal ideals, then M is compact and any map of B into any Hausdorff space can be factored uniquely through φ and A is algebraically and topologically isomorphic to $\Gamma^b(\pi'')$.

b) If M is not compact then A is algebraically and topologically isomorphic to $\Gamma_o(\pi'', \mathfrak{X})$ where $\mathfrak{X}$ is class of compact subsets of M consisting of finite unions of images under φ of the sets in (ii) . Furthermore A is a $C^b(M)$ module.

See K.H. Hofmann and J. Dauns [5].

The condition that multiplication and scalar multiplication are uniformly continuous on bounded sets means that the additive group of bounded sections does form an algebra and this condition is satisfied if A is a Banach algebra.

We shall now show that any complex C*-algebra A satisfies the conditions of the above theorem, where $B = \text{Prim}(A)$ the set of primitive ideals of A in the hull kernel topology.

4.3.3. Definition. Suppose that in a uniform field $(\pi, \mathfrak{U}), \pi : E \to B$ of vector spaces over a normed field K there is a function $\| \ \| : E \to [0, \infty]$, $x \to \| x \|$, $x \in E$ satisfying the following conditions:

i) $\| x \| = 0$ iff $x = 0(\pi(x))$,

ii) $\| rx \| = | r | \| x \|$, $x \in E$, $r \in K$,

iii) $\| x + y \| \leq \| x \| + \| y \|$, $x \nabla y \in E \nabla E$,

iv) $\{(x, y) ; \| x - y \| < \epsilon\}$, $\epsilon > 0$ are a basis for $\mathfrak{U}$ then $(\pi, \mathfrak{U})$ is called a normed field of K vector spaces.

Assume in addition: $(\pi, \mathfrak{U})$ is a uniform field of rings, $K = \mathbb{R}$ or $\mathbb{C}$, scalar multiplication $K \times E \to E$ is continuous and each stalk $\pi^{-1}(b)$, $b \in B$ is a Banach algebra in the induced norm, then $(\pi, \mathfrak{U})$ is called a uniform field of Banach algebras and written as $(\pi, \| \ \|)$. Since both $\Gamma(\pi)$ and $\Gamma^b(\pi)$ are complete and since the usual topology on $\Gamma^b(\pi)$ is given by the Banach algebra norm
$\| \sigma \| = \sup \{ \| \sigma(b) \| ; b \in B\}$ for $\sigma \in \Gamma^b(\pi)$, in this case $\Gamma^b(\pi)$ will be viewed as a Banach algebra. If in addition there is an involution $E \to E$, $x \to x^*$ with $\pi(x^*) = \pi(x)$ and $\| x^* \| = \| x \|$ it is called a uniform field of star algebras. If $\| x^*x \|^2 = \| x \|$, $\forall x \in E$ it is called a uniform field of C*-algebras and then $\Gamma^b(\pi)$ is a C*-algebra.

4.3.4. Proposition. Let D be an algebra over the field K of real or complex numbers and A the algebra obtained by adjoining an identity e so that a partial order with positive cone A^+ is defined on A satisfying the following conditions:

i) If $a \geq 0$ then $(e + a)^{-\frac{+1}{2}}$ and a commute and $(e + a)^{\frac{\pm 1}{2}} \geq 0$;

ii) If $0 \leq a, b$ then $0 \leq bab$;

iii) $\{a ; 0 \leq a \leq 1\}$ is bounded;

iv) If p, q are nets of positive elements such that $pq^2p \to 0$ then $pq \to 0$ and $qp \to 0$;

v) If I is an ideal then linear combinations of elements of $I^+ = A^+ \cap I$ are dense in I .

vi) If $u \to 0$ is a net of positive elements then $\{a ; 0 \leq a \leq u\}$ is eventually in every neighbourhood of zero.

If now B is a set of prime ideals of D such that every proper closed ideal of D is contained in some member of B then D, B has approximate bounded partitions of unity.

See J. Dauns and K.H. Hofmann [5] .

We shall now show that a C*-algebra A with a positive cone $A^+ = \{a \in A ; a = d^*d$ for some $d \in A\}$ satisfies the conditions of Proposition 4.3.4.

4.3.5. Lemma. The partial order of a C*-algebra A with an identity satisfies conditions (i) to (vi) of Proposition 4.3.4.

Proof: (i) Since $e \in A^+$ and any $a \in A^+$ generate a commutative C*-algebra which is isometrically-*-isomorphic to $C(\Phi_A)$ where Φ_A is compact Hausdorff, $(e + a)^{\pm\frac{1}{2}}$ exists, is positive and commutes with a.

(ii) If $a > 0$ then for any $c \in A$, $0 \leq c^*ac$ (See J. Dixmier [6]).

(iii) $\forall\, a, c \in A^+$ if $0 \leq a \leq c$ then $\| a \| \leq \| c \|$. Hence if $0 \leq a \leq e$ then $\| a \| \leq 1$.

(iv) If $0 \leq a, c$ then $ac^2 a = (ac)(ac)^*$ hence $\| ac^2 a \| = \| ac \|^2$.

(v) It will be shown that any complex linear combinations from $I^+ = A^+ \cap I$ are dense in $\overline{I}$.
$\overline{I}^* = \overline{I} \Rightarrow I + I^* \subset \overline{I}$, $\forall\, a \in \overline{I}$, $a = z_1 + iz_2$ where z_1, z_2 are hermitian. Every such hermitian element of $\overline{I}$ in turn is a difference of two positive elements of $\overline{I}$. Thus $\forall\, a \in \overline{I}$, a is a complex linear combination of positive elements of $\overline{I}$. It follows from J. Dixmier [6] that there exists a net $\{u_\lambda\} \subset I^+$ with $0 \leq u_\lambda \leq 1$, $\| u_\lambda \| \leq 1$ $\forall\, \lambda$ having the property that for any $a \in \overline{I}^+$, $\| u_\lambda a - a \| \to 0$ and $\| au_\lambda - a \| \to 0$.
Thus $\| u_\lambda a u_\lambda - a \| \leq \| u_\lambda a - a \| \, \| u_\lambda \| + \| au_\lambda - a \| \to 0$.
By (ii) $\{u_\lambda a u_\lambda\} \subset I^+$.

(vi) Follows as in the proof of (iii) .

We shall now prove the generalised Gelfand-Naimark theorem which shows us that any C*-algebra can be represented as a full set of continuous functions.

4.3.6.Theorem. A a C*-algebra, $B = \mathrm{Prim}(A)$ its structure space with the hull kernel topology Q and $\varphi : (B, Q) \to (M, T)$ the complete regularisation of B where M consists of closed ideals m of A such that $m = \cap\, \varphi^{-1}(m)$. Denote by $\mathfrak{X}$ the family of all T-compact subsets of M of the form $\{m \in M ; \| a + m \| \geq \varepsilon \}$, $a \in A$, $\epsilon > 0$. Let $(\pi'' , \| \ \|)$, $\pi'' : E'' \to (M, T)$ be the uniform field of C*-algebras obtained by first forming the canonical field from A, M and then enlarging the weak-star topology T* of M up to T . For each $a \in A$, $\tilde{a}$ is the map $\tilde{a} : M \to E''$ defined by $\tilde{a}(m) = a + m \in A/m$, then the map $A \to \Gamma_0(\pi'', \mathfrak{X})$, from $a \to \tilde{a}$ is an isometric-*-isomorphism. In particular if M is T compact then $M \in \mathfrak{X}$ and A is isometrically-*-isomorphic to $\Gamma(\pi'', \mathfrak{X})$.

Proof: We shall show that the conditions of Theorem 4.3.2. are satisfied.

i) As every proper closed ideal of A is contained in a member of B (i) follows from Lemma 4.3.5. and Proposition 4.3.4.

ii) If $U = \{a \in A \; ; \; \| a \| < \epsilon\}$ then $a \in b + U$ for $b \in B$ iff $\| \hat{a}(b) \| < \epsilon$. Therefore by Proposition 2.1.11. $\{b \in B \; ; \; \| \hat{a}(b) \| \geqslant \epsilon\}$ is compact.

iii) If $U = \{a \in A \; ; \; \| a \| < \epsilon\}$ then

$\cap \{b + U \; ; \; b \in B\} = \{a \in A \; ; \; \| \hat{a}(b) \| < \epsilon \, , \; \forall \, b \in B\}$.

By Proposition 2.1.7. $\cap \{b + U \; ; \; b \in B\} = U$. Hence (iii).

The fields π and π'' associated with A, B are uniform fields of Banach algebras, hence $\Gamma^b(\pi)$ and $\Gamma^b(\pi'')$ are Banach algebras. By Proposition 2.1.7. the map $A \to \hat{A}$ is an isometry. Since for $a \in A$, $\| \tilde{a} \| = \sup \{\| \tilde{a}(m) \| \; ; \; m \in M\}$ and $\| \tilde{a}(m) \| = \sup \{\| \hat{a}(b) \| , \; b \in B \, , \; b \supset m\}$ it follows that $\| \tilde{a} \| = \| a \|$.

Hence the result follows by Theorem 4.3.2.

In the case that A is commutative then we have that A is isometrically-*-isomorphic to the space of continuous complex valued functions defined on Prim(A).

Notes and Remarks.

The concept of a uniform field, Theorem 4.1.4, Theorem 4.2.4, Theorem 4.3.2, and the representation theorem, (Theorem 4.3.6.) are due to J. Dauns and K.H. Hofmann [5] and K.H. Hofmann [12] .

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